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Howe Correspondence for Real Unitary Groups

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Roger Howe proved that for any reductive dual pair (G, G') in the symplectic group $Sp(2n, \mathbb{R})$, there is a one-to-one correspondence of irreducible admissible representations of some two-fold covers of G and G'. We determine this correspondence explicitly for dual pairs of the form $(U(p, q), U(r, s))$ with $r+s=p+q$, and describe it in terms of Langlands parameters. In this case, the correspondence may be understood in a natural way as a correspondence of representations of the linear groups, rather than the appropriate covers. We show that every irreducible admissible representation of $U(p, q)$ occurs in the correspondence with precisely one unitary group of equal rank. This result verifies a conjecture of Harris, Kudla, and Sweet, who investigated the correspondence for unitary groups of equal size over p-adic fields.

The correspondence of discrete series representations was determined by J.-S. Li. For induced representations, the correspondence is obtained in a natural way from the corresponding discrete series on the respective Levi factors of the parabolic subgroups of $U(p, q)$ and $U(r, s)$. Generalizing a result of Li, we show that under the correspondence representations with nonzero cohomology are matched in an interesting way, with unitarity not necessarily preserved. The proof uses the induction principle which is due to Kudla, and an argument involving K-types and the space of joint harmonics (Howe).

INTRODUCTION

Let G and G' be two groups which are embedded in a larger group H in such a way that they are each other's centralizers in H. Such a pair of groups is called a dual pair. Since G and G' commute inside H,

$$G \cdot G' = \{ gg' \colon g \in G,\ g' \in G' \}$$

is a subgroup of H. If ρ is a representation of H, then by restriction, ρ can be regarded as a representation of $G \cdot G'$, or, via the natural map $G \times G' \to G \cdot G'$, as a representation of $G \times G'$. Every irreducible admissible representation of $G \times G'$ is of the form $\pi \otimes \pi'$, where π and π' are irreducible representations of G and G' respectively.

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Consider the set $\mathscr{R}$ of equivalence classes of irreducible representations $\pi \otimes \pi'$ of $G \times G'$ which can be realized as quotients of ρ; i.e., if ρ is realized on the vector space Y, then (the equivalence class of) $\pi \otimes \pi'$ belongs to $\mathscr{R}$ if there exists a $G \times G'$-invariant subspace X of Y such that the action of $\rho|_{G \times G'}$ on the quotient space Y/X is equivalent to $\pi \otimes \pi'$. If $\pi \otimes \pi' \in \mathscr{R}$, then the irreducible representation π of G is said to correspond to the irreducible representation π' of G'. This correspondence defines a relationship between some of the irreducible representations of G and G'. In general, a representation of G may or may not occur in the correspondence, and it may correspond to none, one, or several different representations of G'. Similarly for a representation of G'.

For any positive integer n, let $\widetilde{Sp}(2n, \mathbb{R})$ be the nontrivial two-fold cover of the symplectic group $Sp(2n, \mathbb{R})$. Suppose (G, G') is a reductive dual pair (i.e., both G and G' are reductive Lie groups) in $Sp = Sp(2n, \mathbb{R})$, and let $\tilde{G}$ and $\tilde{G}'$ be the inverse images of G and G' under the covering map $\widetilde{Sp} \to Sp$. Then $(\tilde{G}, \tilde{G}')$ is a dual pair in $\widetilde{Sp}$, and the oscillator representation [H1] of $\widetilde{Sp}$ gives rise to a dual pair correspondence for $(\tilde{G}, \tilde{G}')$. By a theorem of R. Howe [H3], this correspondence is in fact a bijection between certain sets of irreducible representations of $\tilde{G}$ and $\tilde{G}'$. Of course, if G is a member of two distinct dual pairs (G, G') and (G, G'') in the same symplectic group Sp, then a given representation of $\tilde{G}$ may correspond to both a representation of $\tilde{G}'$ and of $\tilde{G}''$.

We will consider reductive dual pairs of the form $(U(p, q), U(r, s))$ in $Sp = Sp(2(p+q)(r+s), \mathbb{R})$ (see [H1]). Let $\widetilde{Sp}$ be the two-fold metaplectic cover of Sp, and let ω be the oscillator representation of $\widetilde{Sp}$. We denote $G_1 = \tilde{U}(p, q)$ and $G_2 = \tilde{U}(r, s)$ the inverse images by the covering map of $U(p, q)$ and $U(r, s)$ in $\widetilde{Sp}$, and let K_1, K_2, and $\tilde{U} \cong \tilde{U}(n)$ be maximal subgroups of G_1, G_2, and $\widetilde{Sp}$ such that $K_i \subset \tilde{U}$ for $i = 1, 2$. The groups G_1 and G_2 commute inside $\widetilde{Sp}$ as well (see [H3]).

For fixed p, q, r and s, we have a one-to-one correspondence of irreducible admissible representations of G_1 and G_2 as described above. If π and π' are irreducible representations of G_1 and G_2 respectively, and π corresponds to π', we write $\theta_{r,s}(\pi) = \pi'$. If π does not occur in the correspondence, we write $\theta_{r,s}(\pi) = 0$. The cover $\tilde{U}(p, q)$ of $U(p, q)$ depends only on the parity of $r - s$, and is given by the "$\sqrt{det}$-cover" if $r - s$ is odd, and by the trivial two-fold cover if $r - s$ is even. In either case, there is a natural bijection between the sets of equivalence classes of irreducible admissible genuine representations (i.e., which are non-trivial on the kernel of the covering map) of $\tilde{U}(p, q)$ and the admissible dual of the linear group $U(p, q)$. Similarly for $\tilde{U}(r, s)$ and its genuine representations. Consequently, although only genuine representations occur in the dual pair correspondence, there is a natural way to interpret this as a bijection between representations of the groups $U(p, q)$ and $U(r, s)$.

We will consider p and q to be fixed, and let r and s vary, subject to the condition $r+s=p+q$. Notice that the cover of $U(p,q)$ stays the same as r and s vary. Let $\widetilde{U}(p,q)\widehat{\ }_{genuine}$ denote the set of equivalence classes of genuine irreducible admissible representations of $\widetilde{U}(p,q)$. One part of the main result is the following

THEOREM 0.1. *Let $\pi \in \widetilde{U}(p,q)\widehat{\ }_{genuine}$. Then there exist unique values r, s with $r+s=p+q$, such that $\theta_{r,s}(\pi)\neq 0$.*

This result confirms in the real case a conjecture of Kudla, who (with Harris and Sweet) investigated the dual pair correspondence for unitary groups of equal size over non-archimedian fields [HKS]. Moreover, by considering, for every positive integer n, the set of real forms of the complex algebraic group $GL(n,\mathbb{C})$ simultaneously (in the spirit of Vogan L-packets [ABV]), we get a true bijection

$$\bigcup_{p+q=n} U(p,q)\widehat{\ } \leftrightarrow \bigcup_{r+s=n} U(r,s)\widehat{\ }.$$

This is analogous to the result of Adams and Barbasch [AB2], who showed that the Howe correspondence gives rise to a bijection between the genuine irreducible admissible representations of $\widetilde{Sp}(2n,\mathbb{R})$ and those of the groups $SO(p,q)$ with $p+q=2n+1$ and fixed parity of q.

The first step in the proof of Theorem 0.1 is to show that a representation of $\widetilde{U}(p,q)$ may occur in the correspondence with a most one choice of unitary group of the same rank. This is Theorem 2.15 which is proved in Section 2 using a doubling argument due to Kudla and Rallis.

To investigate the correspondence explicitly, we first choose a parametrization of all irreducible admissible representations π of $\widetilde{U}(p,q)$, using the version of the Langlands classification of [V3], and describe the minimal (in the sense of Vogan) K-types of π (Section 3).

The proof of occurrence for any $\pi \in \widetilde{U}(p,q)\widehat{\ }_{genuine}$ combines the known correspondence for discrete series representations, which was proved by J.-S. Li [L], and the induction principle, which is due to Kudla ([K]), and in the version ultimately used in this paper, to Adams and Barbasch ([AB1]). Section 4 is dedicated to stating and proving this version for the case of real unitary groups.

We describe the full correspondence in Section 6, and in Section 5 we prove a series of lemmas needed to apply the induction principle and obtain the explicit matching of Langlands parameters given in Theorem 6.1. The correspondence commutes with real parabolic induction in the sense that representations induced from corresponding discrete series on the Levi factors of real parabolic subgroups correspond to each other. In this way we find that limits of discrete series correspond, and more generally, tempered representations correspond to tempered representations.

THEOREM 0.2. *Let π be a genuine irreducible admissible representation of $\widetilde{U}(p,q)$, with Langlands data (M,σ). Here $M \cong \widetilde{U}(p',q') \times (\mathbb{C}^{\times})^m$ is a real Levi factor, and $\sigma \cong \sigma_0 \otimes \chi$ is a (relative) limit of discrete series of M. Choose r' and s' such that $r'+s' = p'+q'$ and $\sigma_0' = \theta_{r',s'}(\sigma_0) \neq 0$. Then σ_0' is a limit of discrete series. Moreover, $\pi' = \theta_{r'+m,s'+m}(\pi) \neq 0$, and π' has Langlands data (M',σ'). Here $M' \cong \widetilde{U}(r',s') \times (\mathbb{C}^{\times})^m$ and $\sigma' = \sigma_0' \otimes \chi$.*

An important tool is the correspondence of K_1- and K_2-types in the space of joint harmonics ([H3]), which is subordinated to the dual pair correspondence. Howe assigns to each K_i-type a nonnegative integer, the degree, and there is the following close relationship between the two correspondences:

PROPOSITION 0.3 (Howe). *Suppose $\theta_{r,s}(\pi_1) = \pi_2$, σ_1 is a K_1-type occurring in π_1, and that σ_1 is of minimal degree in π_1. Then σ_1 occurs in the space of joint harmonics, and if $\sigma_1 \leftrightarrow \sigma_2$, then σ_2 is a K_2-type of minimal degree in π_2.*

The correspondence of K_i-types in the space of joint harmonics is known for each dual pair, and we have the following fact:

PROPOSITION 0.4. *Every K_1-type occurs in the space of joint harmonics for a unique choice of r and s with $r+s = p+q$.*

It turns out that in the equal rank case, the notions of minimal K-type in the sense of Vogan and K-type of minimal (Howe-)degree coincide in the following sense:

PROPOSITION 0.5. *If $\pi_1 \leftrightarrow \pi_2$, then every minimal K_1-type of π_1 is of minimal degree and corresponds to a minimal K_2-type of π_2 in the space of joint harmonics.*

This has the following consequences:

COROLLARY 0.6. *$\pi_1 \leftrightarrow \pi_2$ implies that π_1 and π_2 have exactly the same number of minimal K_i-types.*

COROLLARY 0.7. *Given $\pi \in \widetilde{U}(p,q)\widehat{\ }_{genuine}$, let σ be a minimal K_1-type of π, and let r and s be such that σ occurs in the space of joint harmonics (as in Proposition 0.4.). Then $\theta_{r,s}(\pi) \neq 0$.*

As an example, we apply Theorem 6.1 to representations of the form $A_{\mathfrak{q}}(\lambda)$, and find that in "most" but not all cases, $A_{\mathfrak{q}}(\lambda)$'s correspond to $A_{\mathfrak{q}}(\lambda)$'s in some natural way, and we give an example where this fails. We conclude that unitarity is not always preserved by the correspondence.

1. PRELIMINARIES

1.1. *The Embedding of the Unitary Groups*

Given non-negative integers k and l, I_k will be the $k \times k$ identity matrix, and $I_{k,l}$ will be the matrix

$$\begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}.$$

Let p, q, r, and s be non-negative integers, and let $n = p+q$, $m = r+s$, and $N = nm$. Let V be a complex vector space of dimension n, equipped with a hermitian form $(\ ,\ )$ with signature p, q, and let W be an m-dimensional space with skew-hermitian form $\langle\ ,\ \rangle$, with signature r, s. Let $\mathbb{W} = V \otimes W \cong \mathbb{C}^N$ be the skew-hermitian space equipped with the form $\langle\!\langle\ ,\ \rangle\!\rangle = (\ ,\ ) \otimes_{\mathbb{C}} \langle\ ,\ \rangle$. Let $U(V)$, $U(W)$, and $U(\mathbb{W})$ be the isometry groups of $(\ ,\ )$, $\langle\ ,\ \rangle$, and $\langle\!\langle\ ,\ \rangle\!\rangle$ respectively. Assume that we have chosen bases of V, W, and $\mathbb{W}$ in such a way that $(\ ,\ )$ is given by the matrix $I_{p,q}$, i.e., $(v_1, v_2) = {}^t\bar{v}_2 I_{p,q} v_1$; $\langle\ ,\ \rangle$ is given by $iI_{r,s}$, and $\langle\!\langle\ ,\ \rangle\!\rangle$ is given by a matrix of the form $J_{\mathbb{W}} = iJ$ with J real diagonal and satisfying $J^2 = I_N$. There is a natural map

$$\mu\colon U(V) \times U(W) \to U(\mathbb{W}) \tag{1.1.1}$$

given by

$$\mu(g, h)(v \otimes w) = (gv) \otimes (hw).$$

For each $g \in U(\mathbb{W})$, write $g = A + iB$, with $A, B \in GL(N, \mathbb{R})$, and let $\nu_J\colon U(\mathbb{W}) \to Sp(\mathbb{W}_{\mathbb{R}})$ be the map defined by

$$\nu_J(g) = \begin{pmatrix} I_N & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} I_N & 0 \\ 0 & J \end{pmatrix}.$$

Here $Sp(\mathbb{W}_{\mathbb{R}}) \cong Sp(2N, \mathbb{R})$ is the isometry group of the symplectic form on $\mathbb{R}^{2N}$ given by $\left(\begin{smallmatrix} 0 & I_N \\ -I_N & 0 \end{smallmatrix}\right)$.Let

$$\alpha_J\colon U(V) \times U(W) \to Sp(\mathbb{W}_{\mathbb{R}}) \tag{1.1.2}$$

be defined by $\alpha_J = \nu_J \circ \mu$. Then α_J imbeds $U(V)$ and $U(W)$ into $Sp(\mathbb{W}_{\mathbb{R}})$ as described in [H1], so that $\alpha_J(U(V))$ and $\alpha_J(U(W))$ are mutual centralizers in $Sp(\mathbb{W}_{\mathbb{R}})$.

Remark 1.1.3. Permuting the basis elements for V and W in the construction of α_J gives rise to an inner automorphism of $Sp(\mathbb{W})$.

1.2. *The Covers*

Let $\widetilde{Sp}$ be the unique nontrivial two-fold cover of the symplectic group $Sp = Sp(2N, \mathbb{R})$. There are two-fold covers $\tilde{U}(p, q)$ and $\tilde{U}(r, s)$ of $U(p, q)$ and $U(r, s)$ and a lifting

$$\tilde{\alpha}\colon \tilde{U}(p, q) \times \tilde{U}(r, s) \to \widetilde{Sp} \tag{1.2.1}$$

of the map α_J of (1.1.2). It will follow from the calculations in Section 1.4 (see Lemma 1.4.5) that $\tilde{U}(p, q)$ is isomorphic to the $det^{(r-s)/2}$-cover of $U(p, q)$

$$= \{ (g, z) \in U(p, q) \times \mathbb{C}^{\times} : z^2 = det(g)^{r-s} \}.$$

Similarly, $\tilde{U}(r, s)$ is isomorphic to the $det^{(p-q)/2}$-cover of $U(r, s)$.

If $r - s$ is even, $\tilde{U}(p, q) \cong U(p, q) \times \mathbb{Z}/2\mathbb{Z}$; otherwise, $\tilde{U}(p, q)$ is isomorphic to the $det^{1/2}$-cover of $U(p, q)$. In either case, the character $det^{(r-s)/2}$ of $\tilde{U}(p, q)$ given by

$$det^{(r-s)/2}(g, z) = z \tag{1.2.2}$$

is genuine. The genuine admissible dual of $\tilde{U}(p, q)$ is therefore given by

$$\tilde{U}(p, q)\widehat{\ }_{genuine} = \{ \pi \otimes det^{(r-s)/2} : \pi \in U(p, q)\widehat{\ } \}.$$

If K is a maximal compact subgroup of $U(p, q)$, then the associated two-fold cover $\tilde{K}$ is a maximal compact subgroup of $\tilde{U}(p, q)$. Unless otherwise specified, we choose for K the set of block diagonal matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ with $A \in U(p)$ and $B \in U(q)$. All $\tilde{K}$-types of a genuine representation of $\tilde{U}(p, q)$ are genuine, and their highest weights (in the usual notation) are of the form

$$(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q)$$

with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_p$, $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_q$, and $a_i, b_j \in \mathbb{Z} + (r-s)/2$. The weight corresponding to the character $det^{(r-s)/2}$ is

$$\Big(\underbrace{\frac{r-s}{2}, ..., \frac{r-s}{2}}_{p}; \underbrace{\frac{r-s}{2}, ..., \frac{r-s}{2}}_{q}\Big).$$

Notation 1.2.3. Let G be a Lie group with maximal compact subgroup K_G. By a *K-type for G* we will mean an irreducible representation of K_G.

1.3. *The Correspondence*

To conserve notation, we use the following

Terminology 1.3.1. Let H be a reductive Lie group with Lie algebra $\mathfrak{h}$ and maximal compact subgroup K_H. A *Harish–Chandra module for H* is an $(\mathfrak{h}, K_H)$-module. If π_1 and π_2 are Harish–Chandra modules for H, then an $(\mathfrak{h}, K_H)$-map $\phi: \pi_1 \to \pi_2$ will often be referred to as an *H-map*.

For ψ a nontrivial unitary additive character of $\mathbb{R}$, let ω_ψ be the associated oscillator representation $\widetilde{Sp} = \widetilde{Sp}(2N, \mathbb{R})$ ([H1]), and let ω_ψ^∞ and ω_ψ^{HC} be the corresponding smooth representation and Harish–Chandra module respectively. Via the map $\tilde{\alpha}$ of (1.2.1), we may regard ω_ψ^∞ as a representation of $\widetilde{U}(p, q) \times \widetilde{U}(r, s)$. Similarly, ω_ψ^{HC} is a Harish–Chandra module for $\widetilde{U}(p, q) \times \widetilde{U}(r, s)$. If $\pi \in \widetilde{U}(p, q)^\frown_{genuine}$ and $\pi' \in \widetilde{U}(r, s)^\frown_{genuine}$, and π_{HC} and π'_{HC} are the associated Harish–Chandra modules, we say that π corresponds to π', or $\theta_{r,s}(\pi) = \pi'$, if there is a $\widetilde{U}(p, q) \times \widetilde{U}(r, s)$-map

$$\phi: \omega_\psi^{HC} \to \pi_{HC} \otimes \pi'_{HC}.$$

According to Theorem 2.1 of [H3], this determines a bijection between certain subsets of $\widetilde{U}(p, q)^\frown_{genuine}$ and $\widetilde{U}(r, s)^\frown_{genuine}$. Notice that every $\widetilde{U}(p, q) \times \widetilde{U}(r, s)$ equivariant map $\phi: \omega_\psi^\infty \to \pi \otimes \pi'$ gives rise to a non-zero map on the corresponding Harish–Chandra modules (by restriction to the space of K-finite vectors).

For ψ a nontrivial additive character of $\mathbb{R}$ and $a \in \mathbb{R}^\times$, define $\psi^a \in \mathbb{R}^\frown$ by

$$\psi^a(y) = \psi(ay).$$

An argument analogous to the one in Chapter 2.II of [MVW] for non-archimedian fields yields the following

LEMMA 1.3.2. (1) *Let $a \in \mathbb{R}^\times$. Then $\omega_\psi \cong \omega_{\psi^a} \Leftrightarrow a > 0$.*

(2) *If ω_ψ^* is the contragredient representation of ω_ψ, then $\omega_{\psi^{-1}} \cong \omega_\psi^*$.*

Consequently, if χ is any nontrivial unitary character of $\mathbb{R}$, then either $\omega_\chi \cong \omega_\psi$ or $\omega_\chi \cong \omega_{\psi^{-1}}$. In the next section we will show (see the proof of Proposition 2.1) that the Howe correspondences for $(U(p,q), U(r,s))$ obtained from ω_ψ and $\omega_{\psi^{-1}}$ are related as follows: For $\pi \in \widetilde{U}(p,q)^{\widehat{}}_{genuine}$,

$$\theta^{\psi}_{r,s}(\pi) = \pi' \Leftrightarrow \theta^{\psi^{-1}}_{r,s}(\pi^*) = \pi'^*. \tag{1.3.3}$$

Note on Notation 1.3.4. In the next section we will make a choice for ψ and then drop it from the notation. Moreover, ω, ω^∞, and the associated Harish–Chandra module ω^{HC} will all be denoted ω. Similarly for representations of unitary groups.

1.4. *The Space of Joint Harmonics*

The following discussion is in [H3]. Let (G, G') be a reductive dual pair in $Sp = Sp(2n, \mathbb{R})$. Let $U \cong U(n)$, K, and K' be maximal compact subgroups of Sp, G, and G' respectively, with $K \cdot K' \subset U$. Let $\widetilde{Sp}$ be the nontrivial two-fold cover of Sp, and denote $\widetilde{G}$, $\widetilde{U}$, $\widetilde{K}$, etc. the corresponding two-fold covers of the subgroups. Let ω be the oscillator representation of $\widetilde{Sp}$. In the Fock model of ω, the space of $\widetilde{U}$-finite vectors of the Harish–Chandra module associated to ω may be identified with the space $\mathscr{P}$ of polynomials on $\mathbb{C}^n$, in such a way that the action of $\widetilde{U}$ preserves the degree of polynomials. Consequently, we may make the following.

Definition 1.4.1. Let σ be a $\widetilde{K}$-type or $\widetilde{K}'$-type occurring in $\mathscr{P}$. The *degree of* σ is the degree of the polynomial of least degree in the σ-isotypic subspace of $\mathscr{P}$.

There is a $\widetilde{K} \times \widetilde{K}'$-invariant subspace $\mathscr{H}$ of $\mathscr{P}$, the *space of joint harmonics*, with the following properties:

Theorem 1.4.2 (Howe). (1) *$\widetilde{K}$ and $\widetilde{K}'$ generate mutual commutants on $\mathscr{H}$, i.e.,*

$$\mathscr{H} \cong \bigoplus_i \sigma_i \otimes \sigma'_i, \tag{1.4.3}$$

where for all i, $\sigma_i \in \widetilde{K}^{\widehat{}}$ and $\sigma'_i \in (\widetilde{K}')^{\widehat{}}$, and the representations σ_i and σ'_i determine each other. Moreover, for all i, σ_i and σ'_i have the same degree.

(*If $\sigma \otimes \sigma'$ is a summand in* (1.4.3), *we say that σ corresponds to σ' in $\mathscr{H}$.*)

(2) *Suppose $\pi \in \widetilde{G}^{\widehat{}}$, $\pi' \in (\widetilde{G}')^{\widehat{}}$, and $\pi \leftrightarrow \pi'$ in the correspondence for the dual pair (G, G'). Let σ be a $\widetilde{K}$-type occurring in π, and suppose that σ is of minimal degree among the $\widetilde{K}$-types of π. Then σ occurs in $\mathscr{H}$, and the $\widetilde{K}'$-type σ' which corresponds to σ in $\mathscr{H}$ is a $\widetilde{K}'$-type of minimal degree in π'.*

Now consider dual pair of the form $(U(p,q), U(r,s))$. For fixed p and q, the degree of a K-type for $\widetilde{U}(p,q)$ will depend on r and s, and so will whether σ occurs in $\mathscr{H}$. Consequently, we will use the following

Terminology 1.4.4. Let σ be a K-type for $\widetilde{U}(p,q)$.

(1) The r, s-degree of σ is the degree of σ for $U(p,q)$ a member of the dual pair $(U(p,q), U(r,s))$.

(2) If σ occurs in the space of joint harmonics for the dual pair $(U(p,q), U(r,s))$, we say that σ is *r, s-harmonic*.

Now fix p, q, r, and s. A direct computation (see also [KaV]) yields the correspondence of K-types in $\mathscr{H}$ as given in the following lemma.

LEMMA 1.4.5. *There is a choice for ψ such that the correspondence of K-types for $\widetilde{U}(p,q)$ and $\widetilde{U}(r,s)$ in the space of joint harmonics for the dual pair $(U(p,q), U(r,s))$ is given as follows*:

(1) *If μ is a K-type for $\widetilde{U}(p,q)$, then μ occurs in $\mathscr{H}$ if and only if the highest weight of μ is of the form*

$$\Bigg(\overbrace{\frac{r-s}{2},\ldots,\frac{r-s}{2}}^{p};\overbrace{\frac{s-r}{2},\ldots,\frac{s-r}{2}}^{q}\Bigg)$$
$$+(a_1,a_2,\ldots,a_k,0,\ldots,0,b_1,\ldots,b_l;c_1,c_2,\ldots,c_m,0,\ldots,0,d_1,\ldots,d_n), \tag{1.4.6a}$$

with $k+n\leqslant r$ and $l+m\leqslant s$. Then $\mu\leftrightarrow\mu'$, where μ' is the K-type for $\widetilde{U}(r,s)$ with highest weight

$$\Bigg(\overbrace{\frac{p-q}{2},\ldots,\frac{p-q}{2}}^{r};\overbrace{\frac{q-p}{2},\ldots,\frac{q-p}{2}}^{s}\Bigg)$$
$$+(a_1,a_2,\ldots,a_k,0,\ldots,0,d_1,\ldots,d_n;c_1,c_2,\ldots,c_m,0,\ldots,0,b_1,\ldots,b_l). \tag{1.4.6b}$$

(2) *If ν is a K-type for $\widetilde{U}(p,q)$ which occurs in the oscillator representation, with highest weight*

$$\Bigg(\underbrace{\frac{r-s}{2},\ldots,\frac{r-s}{2}}_{p};\underbrace{\frac{s-r}{2},\ldots,\frac{s-r}{2}}_{q}\Bigg)+(x_1,x_2,\ldots,x_p;y_1,\ldots,y_q), \tag{1.4.7}$$

then the r, s-degree of v is given by

$$\sum_{i=1}^{p} |x_i| + \sum_{i=1}^{q} |y_i|. \tag{1.4.8}$$

Remark 1.4.9. Formula (1.4.8) makes sense for every K-type, not only those which occur in the oscillator representation. It will be convenient to extend the notion of r, s-degree to all K-types by formal definition using (1.4.7) and (1.4.8).

PROPOSITION 1.4.10. *Suppose μ is a genuine K-type for $\tilde{U}(p, q)$ with highest weight $(x_1, x_2, ..., x_p; y_1, ..., y_q)$. Then there are unique integers r and s such that $r+s=p+q$ and μ is r, s-harmonic.*

Proof. If r and s satisfy $r+s=p+q$, then by Lemma 1.4.5, μ is r, s-harmonic if and only if there are nonnegative integers a and c with $a+(q-c)=r$ such that

$$(x_1, ..., x_p; y_1, ..., y_q) = \left(\frac{r-s}{2}, ..., \frac{r-s}{2}; -\frac{r-s}{2}, ..., -\frac{r-s}{2}\right)$$
$$+ (\alpha_1, ..., \alpha_a, \alpha_{a+1}, ..., \alpha_p; \beta_1, ..., \beta_c, \beta_{c+1}, ..., \beta_q),$$

where $\alpha_a \geqslant 0$, $\alpha_{a+1} \leqslant 0$, $\beta_c \geqslant 0$, and $\beta_{c+1} \leqslant 0$.

Let $n=p+q$. Consider the n-tuple $(x_1, x_2, ..., x_p, -y_q, -y_{q-1}, ..., -y_1)$. Let $(z_1, z_2, ..., z_n)$ be the n-tuple obtained from it by rearranging the entries in non-increasing order. Then $(z_1+(n-1)/2,\ z_2+(n-3)/2, ..., z_n-(n-1)/2)$ is a strictly decreasing n-tuple of integers.

If $z_1+(n-1)/2>0$ let $r=\max\{i\colon z_i+(n-2i+1)/2>0\}$. Otherwise let $r=0$. Let $s=n-r$. Choose nonnegative integers a and d with $a+d=r$ as follows: If $r=0$ let $a=d=0$. Otherwise, pick a and d so that there are a x_i's and d y_j's such that $x_i \geqslant z_r$ and $-y_j \geqslant z_r$. Let $c=q-d$. Then it is easy to check that $x_a \geqslant (r-s)/2 \geqslant x_{a+1}$ and $y_c \geqslant -(r-s)/2 \geqslant y_{c+1}$, so that r, s, a, and c have the required properties.

For uniqueness, suppose $r+s=n$ and $r'+s'=n$ both work, with corresponding α_i, β_i, α'_i, β'_i, a, c, a', and c'. Suppose $r>r'$, so that $(r-s)/2 > (r'-s')/2$. Then $\alpha'_{a'} > \alpha_a \geqslant 0$ implies that $a' \geqslant a$, and similarly, $c' \leqslant c$. But then $r' = a'+q-c' \geqslant a+q-c=r$, contradicting $r>r'$. Similarly if $r<r'$. ∎

2. UNIQUENESS OF OCCURRENCE

The main results of this section (Proposition 2.1, Lemma 2.8, and Theorem 2.9) and techniques of proof are essentially standard (cf. [AB2] and [MVW]).

The groups $U(p, q)$ and $U(r, s)$ are naturally isomorphic, and there is a corresponding isomorphism τ of the covers. To conserve notation, we will often identify $\widetilde{U}(p, q)$ with $\widetilde{U}(q, p)$ without referring to τ.

Fix p, q, r, and s, and consider the dual pairs $(U(p, q), U(r, s))$ and $(U(p, q), U(s, r))$. Notice that the covers which $U(p, q)$ inherits as a member of the two dual pairs are isomorphic. For $\pi \in \widetilde{U}(p, q)^{\widehat{}}_{genuine}$ and $\pi' \in \widetilde{U}(r, s)^{\widehat{}}_{genuine}$, let π^* and π'^* denote the contragredient representations of π and π' respectively.

Proposition 2.1. *Suppose* $\pi \in \widetilde{U}(p, q)^{\widehat{}}_{genuine}$, $\pi' \in \widetilde{U}(r, s)^{\widehat{}}_{genuine}$, *and* $\theta_{r, s}(\pi) = \pi'$. *Then* $\theta_{s, r}(\pi^*) = \pi'^*$.

Proof. Let $n = p + q$, $m = r + s$, and $N = nm$. Realize the dual pairs $(U(p, q), U(r, s))$ and $(U(p, q), U(s, r))$ in $Sp(2N, \mathbb{R})$ as in Section 1.1. Let J and α_J correspond to the pair $(U(p, q), U(r, s))$. Using Remark 1.1.3, we may assume that the skew-hermitian form corresponding to $U(s, r)$ is given by the matrix $-iI_{r,s}$. Consequently, $U(p, q) \times U(s, r)$ is mapped to $Sp(\mathbb{W}_{\mathbb{R}})$ via the map α_{-J}. We have the following commutative diagram:

$$\begin{array}{ccc} U(p, q) \times U(r, s) & \xrightarrow{\alpha_J} & Sp(\mathbb{W}_{\mathbb{R}}) \\ {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ U(p, q) \times U(s, r) & \xrightarrow{\alpha_{-J}} & Sp(\mathbb{W}_{\mathbb{R}}), \end{array} \tag{2.2}$$

where σ is the outer automorphism of $Sp(2N, \mathbb{R})$ given by conjugation by $I_{N, N}$. It follows from the definition of α_J and α_{-J} that for $g \in U(p, q)$ and $g' \in U(r, s)$,

$$\sigma \circ \alpha_J(g, g') = \alpha_{-J}(g, g') = \alpha_J(\bar{g}, \bar{g}'), \tag{2.3}$$

where $\bar{\cdot}$ denotes complex conjugation of the matrix entries.

There is a lifting of σ to an automorphism $\tilde{\sigma}$ of $\widetilde{Sp}(2N, \mathbb{R})$ ([R]), and this induces automorphisms σ_1 and σ_2 of $\widetilde{U}(p, q)$ and $\widetilde{U}(r, s)$. The maps σ_1 and σ_2 are given by $\sigma_i(g, z) = (\bar{g}, \bar{z})$. Let ω^σ be the representation of $\widetilde{Sp}(2N, \mathbb{R})$ defined by

$$\omega^\sigma(g) = \omega(\tilde{\sigma}(g)). \tag{2.4}$$

We have that $\omega^\sigma \cong \omega^*$. For π and π' representations of $\widetilde{U}(p, q)$ and $\widetilde{U}(r, s)$ respectively, define π^{σ_1} and π'^{σ_2} analogously. Then we have that

$$\omega_{p, q, s, r}|_{\widetilde{U}(p, q) \times \widetilde{U}(s, r)} \cong \omega^\sigma_{p, q, r, s}|_{\widetilde{U}(p, q) \times \widetilde{U}(r, s)}. \tag{2.5}$$

Notice that statement (2.5) also holds in the setting of Harish–Chandra modules. So the proposition follows if

$$\pi^{\sigma_1} \cong \pi^* \qquad \text{and} \qquad \pi'^{\sigma_2} \cong \pi'^*. \tag{2.6}$$

LEMMA 2.7. *Let $\rho \in U(p,q)\widehat{\ }$, and define $\bar{\rho} \in U(p,q)\widehat{\ }$ by $\bar{\rho}(g) = \rho(\bar{g})$. Then $\bar{\rho} \cong \rho^*$.*

Proof. A straightforward calculation shows that the global characters of $\bar{\rho}$ and ρ^* agree. ∎

Let $\pi \in \widetilde{U}(p,q)\widehat{\ }_{genuine}$, $\pi \cong \rho \otimes det^{(r-s)/2}$ for some $\rho \in U(p,q)\widehat{\ }$. Then $\pi^* \cong \rho^* \otimes det^{-(r-s)/2}$, so that

$$\begin{aligned}\pi^{\sigma_1}(g,z) &= \pi(\bar{g},\bar{z}) = \bar{z} \cdot \rho(\bar{g}) = z^{-1} \cdot \bar{\rho}(g) \\ &= (\bar{\rho} \otimes det^{-(r-s)/2})(g,z) \overset{\text{Lemma 2.7}}{\cong} \pi^*(g,z),\end{aligned}$$

and the proposition is proved. ∎

For r, s, r', s' nonnegative integers, there is a natural embedding

$$\varphi\colon U(r,s) \times U(r',s') \hookrightarrow U(r+r', s+s'),$$

which lifts to a map $\tilde{\varphi}$ of the covers, with kernel of order 2.

LEMMA 2.8. *For nonnegative integers p, q, r, s, r', s', with $r+s \equiv r'+s'$ (mod 2), let $\omega_{p,q,r,s}$ be the oscillator representation for the dual pair $(U(p,q), U(r,s))$, and let $\omega_{p,q,r',s'}$ and $\omega_{p,q,r+r',s+s'}$ be defined analogously. Then*

(a) $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'} |_{\widetilde{U}(r,s) \times \widetilde{U}(r',s')} \cong \omega_{p,q,r+r',s+s'} |_{\widetilde{U}(r,s) \times \widetilde{U}(r',s')}$;

(b) *the diagonal action of $\widetilde{U}(p,q)$ in $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'}$ factors to $U(p,q)$, and $\omega_{p,q,r,s} \otimes \omega_{p,q,r',s'} \cong \omega_{p,q,r+r',s+s'}$ as representations of $U(p,q)$.*

The analogous statements in the setting of Harish–Chandra modules hold as well.

Proof. Let V be a hermitian space with signature p, q, and let W_1 and W_2 be skew-hermitian spaces with signature r, s and r', s' respectively. Let

$$V_1 \cong V_2 \cong V, \qquad \mathbb{W}_i = V_i \otimes W_i \qquad \text{for} \quad i = 1, 2, \qquad \text{and}$$

$$\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2 \cong V \otimes (W_1 + W_2).$$

Then (a) follows from the fact [R] that

$$\omega|_{\widetilde{Sp}(\mathbb{W}_1)\times\widetilde{Sp}(\mathbb{W}_2)} \cong \omega_1 \otimes \omega_2,$$

where ω, ω_1, and ω_2 are the oscillator representations of $\widetilde{Sp}(\mathbb{W})$, $\widetilde{Sp}(\mathbb{W}_1)$, and $\widetilde{Sp}(\mathbb{W}_2)$ respectively.

For (b) we have the commutative diagram

$$\begin{tikzcd} Sp(\mathbb{W}_1)\times Sp(\mathbb{W}_2) \arrow[r] & Sp(\mathbb{W}) \\ U(V_1) \arrow[u, "\beta"] \arrow[r, "\cong"] & U(V), \arrow[u] \end{tikzcd}$$

where the map β is given by

$$U(V_1) \overset{\Delta}{\hookrightarrow} U(V_1)\times U(V_2) \hookrightarrow Sp(\mathbb{W}_1)\times Sp(\mathbb{W}_2).$$

Here Δ denotes the diagonal imbedding. The diagram lifts to the covers, which implies

$$(\omega_1\otimes\omega_2)|_{\tilde{U}(V_1)} \cong \omega|_{\tilde{U}(V)}.$$

Part (b) follows. ∎

THEOREM 2.9. *Let* $\pi \in \tilde{U}(p,q)^{\widehat{}}_{genuine}$, *and suppose* $\theta_{r,s}(\pi)\neq 0$ *and* $\theta_{r',s'}(\pi)\neq 0$, *where* $r+s=r'+s'=p+q$. *Then* $r=r'$ *and* $s=s'$.

Proof. By Proposition 2.1, $\theta_{r',s'}(\pi)\neq 0 \Rightarrow \theta_{s',r'}(\pi^*)\neq 0$. So we have

$$\begin{aligned} &\theta_{r,s}(\pi)\neq 0 \qquad \text{and} \qquad \theta_{r',s'}(\pi)\neq 0 \\ &\quad\Rightarrow Hom_{\tilde{U}(p,q)\times\tilde{U}(p,q)}(\omega_{p,q,r,s}\otimes\omega_{p,q,s',r'}, \pi\otimes\pi^*)\neq 0 \\ &\quad\Rightarrow Hom_{U(p,q)}(\omega_{p,q,r+s',s+r'}, \pi\otimes\pi^*)\neq 0 \qquad \text{(by Lemma 2.8)} \\ &\quad\Rightarrow Hom_{U(p,q)}(\omega_{p,q,r+s',s+r'}, \mathbb{1})\neq 0 \\ &\quad\Rightarrow \theta_{r+s',s+r'}(\mathbb{1})\neq 0. \end{aligned}$$

(Here the *Hom* spaces are those in the category of Harish–Chandra modules.)

The theorem now follows immediately from

LEMMA 2.10. *Let* R, S *be any nonnegative integers such that* $R+S$ *is even, and suppose* $\theta_{R,S}(\mathbb{1})\neq 0$. *Then* $R=S$ *or* $R\geqslant p+q$ *and* $S\geqslant p+q$.

Proof. This follows from the fact that if $\theta_{R,S}(\mathbb{1}) \neq 0$ then the trivial $(U(p) \times U(q))$-type is R, S-harmonic, using Lemma 1.4.5. ▮

3. GENUINE REPRESENTATIONS OF $\tilde{U}(p, q)$ (LANGLANDS PARAMETERS) AND MINIMAL K-TYPES

3.1. *Langlands Parameters*

The following description of $\tilde{U}(p, q)\widehat{\ }_{genuine}$ uses the version of the Langlands classification of [V3] and [KV].

Fix nonnegative integers p and q, and let $N = p + q$. For the remainder of this section, we will use the following notation: if m and n are any nonnegative integers, $\tilde{U}(m, n)$ will denote the $det^{N/2}$-cover of $U(m, n)$. Let $G = \tilde{U}(p, q)$, and let K be the inverse image by the covering map of the usual block diagonal compact subgroup of $U(p, q)$. Let $\mathfrak{g}_0$, $\mathfrak{g}$ and $\mathfrak{k}_0$, $\mathfrak{k}$ be the real, complexified Lie algebras of G and K, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition, and let T be the compact Cartan subgroup of G with real and complexified Lie algebras $\mathfrak{t}_0$ and $\mathfrak{t}$ consisting of diagonal matrices. The compact roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ are

$$\Delta_c = \{e_i - e_j : 1 \leqslant i, j \leqslant p\} \cup \{f_i - f_j : 1 \leqslant i, j \leqslant q\}, \tag{3.1.1a}$$

and the noncompact roots are

$$\Delta_n = \{\pm(e_i - f_j) : 1 \leqslant i \leqslant p \text{ and } 1 \leqslant j \leqslant q\}. \tag{3.1.1b}$$

Here $e_i \in \mathfrak{t}^*$ denotes the element with 1 in the ith coordinate, and all other coordinates 0, and f_j is the element with 1 in the $(p + j)$th coordinate, and all others 0.

Discrete Series and Limits of Discrete Series

The genuine discrete series representations π of $\tilde{U}(p, q)$ are parametrized by Harish–Chandra parameters, of the form

$$\lambda = (a_1, a_2, ..., a_p; b_1, ..., b_q) \in i\mathfrak{t}_0^*, \tag{3.1.2}$$

where $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$ and $a_i \neq b_j$ for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$, and $a_1 > a_2 > \cdots > a_p$, as well as $b_1 > b_2 > \cdots > b_q$. Let Ψ be the positive root system determined by λ, i.e.,

$$\alpha \in \Psi \Leftrightarrow \langle \alpha, \lambda \rangle > 0. \tag{3.1.3}$$

Then π has a *minimal* (or *lowest*, Def. 5.4.18 of [V2]) K-type with highest weight $\Lambda = \lambda + \rho_n - \rho_c$, where ρ_n and ρ_c are one half the sums of the noncompact and compact roots in Ψ respectively. The infinitesimal character of π is τ_λ, where $\lambda \mapsto \tau_\lambda$ is the Harish–Chandra map. We will abuse notation and parametrize infinitesimal characters by elements of the duals of Cartan subalgebras without reference to the Harish–Chandra map.

Genuine limit of discrete series representations π of G are parametrized by pairs (λ, Ψ), where $\lambda \in i\mathfrak{t}_0^*$ is of the form

$$\lambda = (\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}; \underbrace{a_1, ..., a_1}_{l_1}, \underbrace{a_2, ..., a_2}_{l_2}, ..., \underbrace{a_x, ..., a_x}_{l_x}), \tag{3.1.4}$$

with $a_i \in \mathbb{Z} + \frac{1}{2}$, $a_1 > \cdots > a_x$, $p = \sum_i k_i$, and $q = \sum_i l_i$, and Ψ is a system of positive roots. Moreover, λ and Ψ satisfy the following conditions: $|k_i - l_i| \leqslant 1$ for $1 \leqslant i \leqslant x$, Ψ contains $\{e_i - e_j : i < j\}$ and $\{f_i - f_j : i < j\}$; for any root $\alpha \in \Psi$, $\langle \lambda, \alpha \rangle \geqslant 0$, and if α is a *simple* root in Ψ with $\langle \alpha, \lambda \rangle = 0$ then α is noncompact. This last property is condition F-1 of [V3].

As for the discrete series, π has a unique lowest K-type with highest weight $\lambda + \rho_n - \rho_c$, where ρ_n and ρ_c are as above, and the infinitesimal character of π is λ.

Cartan Subgroups and Standard Representations

For $0 \leqslant k \leqslant \min\{p, q\}$, choose a theta stable Cartan subgroup $H_k = T_k A_k$ of G with Lie algebra $\mathfrak{h}_0^k = \mathfrak{t}_0^k + \mathfrak{a}_0^k$ such that $\mathfrak{t}_0^k \subseteq \mathfrak{t}_0$, $\mathfrak{a}_0^k \subseteq \mathfrak{p}_0$, $T_k \cong (S^1)^{N-k}$, and $A_k \cong \mathbb{R}^k$. Up to K-conjugacy, these are all the theta stable Cartan subgroups of G. For fixed k, let $M = Z_G(A_k)$. Then

$$M \cong \tilde{U}(p-k, q-k) \times (\mathbb{C}^\times)^k. \tag{3.1.5}$$

In the case when $p = q = k$, $\tilde{U}(0, 0)$ denotes the kernel of the covering map, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Given a limit of discrete series representation $\sigma = \sigma(\lambda, \Psi)$ of $\tilde{U}(p-k, q-k)$ and a character

$$\chi = \chi_1 \otimes \cdots \otimes \chi_k \in ((\mathbb{C}^\times)^k)\widehat{\ }$$

with differential $d\chi$, choose a real parabolic subgroup $P = MN$ of G such that

$$Re\langle \alpha, d\chi \rangle \geqslant 0 \qquad \text{for all} \quad \alpha \in \Delta(\mathfrak{n}, \mathfrak{a}). \tag{3.1.6}$$

Let $\bar{\gamma} \in (\mathfrak{h}^k)^*$ be given by $\bar{\gamma} = \lambda + d\chi$, and let γ be the pair $(\Psi, \bar{\gamma})$. Then the *standard representation with parameter* γ is

$$I(\gamma) = Ind_{MN}^G(\sigma \otimes \chi \otimes \mathbb{1}). \tag{3.1.7}$$

We use normalized induction. For $1 \leqslant i \leqslant k$, write $\chi_i = \chi_{\mu_i, \nu_i}$, where $\chi_{\mu_i, \nu_i}(re^{i\theta}) = r^{\nu_i} e^{i\mu_i\theta}$. The representation $I(\gamma)$ has finite length, and infinitesimal character $\bar{\gamma}$, or (via the Cayley transform c: $\mathfrak{h}^* \to \mathfrak{t}^*$ defined below) $\bar{\gamma}_c$, where

$$\bar{\gamma}_c = c(\bar{\gamma}) = \lambda + (\tfrac{1}{2}(\mu_1 + \nu_1), \tfrac{1}{2}(\mu_2 + \nu_2), ..., \tfrac{1}{2}(\mu_k + \nu_k);$$
$$\tfrac{1}{2}(\mu_1 - \nu_1), \tfrac{1}{2}(\mu_2 - \nu_2), ..., \tfrac{1}{2}(\mu_k - \nu_k)) \in \mathfrak{t}^*. \tag{3.1.8}$$

If for all $i \leqslant k$, we have that $\nu_i = 0 \Rightarrow \mu_i \in 2\mathbb{Z}$ (this is condition F-2 in [V3]), then $I(\gamma)$ has a unique irreducible quotient $\bar{I}(\gamma)$.

Every representation in $\widetilde{U}(p, q)\widehat{\ }_{genuine}$ is equivalent to such a quotient $\bar{I}(\gamma)$, and two such quotients $\bar{I}(\gamma_1)$ and $\bar{I}(\gamma_2)$ are equivalent if and only if the theta stable Cartan subgroups coincide and γ_2 is obtained from γ_1 by permuting the coordinates and by replacing of the ν_i in χ by $-\nu_i$.

Every representation in $\widetilde{U}(p, q)\widehat{\ }_{genuine}$ may also be obtained as an irreducible quotient of some $I(\gamma_d)$, where $I(\gamma_d) = Ind_{MN}^{G}(\sigma \otimes \chi \otimes \mathbb{1})$, with as before $M \cong (\widetilde{U}(p-k, q-k) \times (\mathbb{C}^\times)^k$, but σ is a discrete series representation of $\widetilde{U}(p-k, q-k)$. In general, χ will then not satisfy condition F-2, and $I(\gamma_d)$ will be a sum of representations, each of which has a unique irreducible quotient.

3.2. *Minimal K-Types*

All minimal K-types of a standard representation have multiplicity one and coincide with the minimal K-types of the irreducible quotients.

In order to compute the minimal K-types of a standard representation $I(\gamma)$ of G, we want to realize its underlying $(\mathfrak{g}, K)$-module $X(\gamma)$ as cohomologically induced from a θ-stable subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$. The Levi subgroup L of G with Lie algebra $\mathfrak{l}$ will be the inverse image by the covering map in G of a group of the form $\prod_i U(m_i, n_i)$. There is a natural map

$$\phi\colon \prod_i \widetilde{U}(m_i, n_i) \to L \tag{3.2.1}$$

and a bijection

$$L\widehat{\ }_{genuine} \leftrightarrow \prod_i \widetilde{U}(m_i, n_i)\widehat{\ }_{genuine}. \tag{3.2.2}$$

We need the notion of *fine* K-types (see Definition 4.3.9 in [V2]) of any L whicn is quasisplit. Recall that if L is of the above form then L is quasisplit if and only if $|m_i - n_i| \leqslant 1$. Using the definition, a direct calculation yields the following result.

LEMMA 3.2.3. (1) *Let μ be a K-type of $\widetilde{U}(n, n)$. Then μ is fine if and only if the highest weight of μ is of the form* $(a, ..., a; b, ..., b)$, *where* $|a-b| \leqslant 1$.

(2) *Let μ be a K-type of $\widetilde{U}(m, n)$ with $|m-n|=1$. Then μ is fine if and only if the highest weight of μ is of the form* $(a, ..., a; a, ..., a)$.

LEMMA 3.2.4. (1) *Let $\pi = Ind_{\widetilde{U}(0,0) \times (\mathbb{C}^\times)^k \cdot N}^{\widetilde{U}(k,k)}(sgn \otimes \xi \otimes \mathbb{1})$ be a genuine principal series representation of $\widetilde{U}(k, k)$, with ξ given by*

$$\xi(r_1 e^{i\theta_1}, ..., r_k e^{i\theta_k}) = r_1^{\nu_1} e^{in\theta_1} \cdots r_k^{\nu_k} e^{in\theta_k}.$$

Then the minimal K-types of π are the one-dimensional K-types with weights μ_i of the form

$$\mu_i = \left(\frac{n}{2}, ..., \frac{n}{2}; \frac{n}{2}, ..., \frac{n}{2}\right) + (a, ..., a; -a, ..., -a),$$

$$\text{where} \quad a = \begin{cases} 0 & \text{if} \quad n \in 2\mathbb{Z}+N, \\ \pm\frac{1}{2} & \text{if} \quad n \in 2\mathbb{Z}+N+1. \end{cases}$$

(2) *Let $n \in 2\mathbb{Z}+N$, and let $\pi = Ind_{\widetilde{U}(1) \times (\mathbb{C}^\times)^k \cdot N}^{\widetilde{U}(k+1,k)}(\zeta \otimes \xi \otimes \mathbb{1})$ be a genuine principal series representation of $\widetilde{U}(k+1, k)$, with ξ as in part* (1) *and ζ given by*

$$\begin{cases} \zeta(e^{i\psi}, \varepsilon) = \varepsilon e^{i(n/2)\psi} & \text{if } N \text{ is even} \\ \zeta(2^{2i\psi}, e^{i\psi}) = e^{in\psi} & \text{if } N \text{ is odd.} \end{cases}$$

Then π has a unique minimal (*one-dimensional*) *K-type with weight* $(n/2, ..., n/2; n/2, ..., n/2)$.

Proof. (1) Using Frobenius Reciprocity it is easy to check that the μ_i above are precisely the fine K-types of π. Moreover, since the weights of a given K-type differ by a sum of roots, every K-type of π has a highest weight of the form

$$(a_1, ..., a_k; b_1, ..., b_k) \qquad \text{with} \qquad \sum_i a_i + \sum_i b_i = kn. \tag{3.2.5}$$

Since the K-types of π are independent of the ν_i, we may assume (by picking ν_i imaginary for all i) that π is unitary. By Theorem 4.3.17 of [V2], every irreducible component of π contains a fine K-type. By Lemma 8.8 of [V1], the minimal K-types of an irreducible representation are precisely those which are *lambda-lowest* (Def. 5.4. of [V2]). So (since it is easy to check that all μ_i have the same (Vogan)-norm) it is sufficient to show that the fine K-types of π coincide with the lambda-lowest ones.

Let μ be the highest weight of a K-type of π, and let $\lambda=\lambda(\mu)$ be as in the definition of "lambda-lowest". Then it follows from the definition and using (3.2.5) that $\lambda=(x_1, ..., x_k; y_1, ..., y_k)$ with $\sum_i x_i+\sum_i y_i=kn$. Subject to this constraint, $\langle\lambda, \lambda\rangle$ is strictly minimized if λ is a scalar. Using Definition 5.3.24 and Cor. 5.4.14 of [V2], it follows that $\lambda(\mu)$ is a scalar if and only if μ is fine, and (1) is proved.

(2) is analogous. ∎

LEMMA 3.2.6. *Let $I(\gamma)$ be a standard representation of G with $\gamma=(\Psi, \bar{\gamma})$ and $\bar{\gamma}=\lambda+d\chi$. Suppose λ is given by* (3.1.4). *Let*

$$\lambda_d=(a_{i_1}, ..., a_{i_r}; a_{i_{r+1}}, ..., a_{i_{r+s}}),$$

where $\{a_{i_1}, ..., a_{i_r}\}$ lists the a_i with the property that $k_i>\ell_i$, and $\{a_{i_{r+1}}, ..., a_{i_{r+s}}\}$ lists the a_i such that $k_i<\ell_i$. Let $I(\gamma_d)$ be the standard representation induced from discrete series, with associated Cartan subgroup $H_{p-r}(=H_{q-s})$ and $\gamma_d=(\Psi_d, \bar{\gamma}_d)$ given by $\bar{\gamma}_d=\lambda_d+d\chi_d$, where $\chi_d=\chi\otimes\bigotimes_{i=1}^{x}(\chi_{2a_i,0})^{m_i}$ with $m_i=\min\{k_i, \ell_i\}$, and $\Psi_d=\Psi_d(\lambda_d)$ (which is uniquely determined). Then $I(\gamma)$ is a direct summand of $I(\gamma_d)$.

Proof. Using Induction by Stages it is sufficient to consider the case where $I(\gamma)$ is a limit of discrete series representation. In this case, the result follows from Theorem 14.71 of [Kn] by induction on $k=p-r$. ∎

Let $L=\prod_i \widetilde{U}(m_i, n_i)\subset G$ be a Levi subgroup which is quasisplit, and let $\delta\in i\mathfrak{t}_0^*$. Then δ will be called a *fine weight of L* if δ is of the form

$$\sum_i (\underbrace{a_i, ..., a_i}_{m_i}; \underbrace{-a_i, ..., a_i}_{n_i}) \qquad \text{with} \qquad \begin{cases} a_i=0 & \text{if } m_i\neq n_i, \\ a_i\in\{0, \pm\frac{1}{2}\} & \text{if } m_i=n_i. \end{cases}$$

PROPOSITION 3.2.7. *Let $\pi=\bar{I}(\gamma)$ be an irreducible genuine representation of G, with infinitesimal character $\bar{\gamma}_c$ as in* (3.1.8), *and associated Cartan subalgebra $H=TA$. Write $\bar{\gamma}_c=\lambda+\mu+\nu$, where $\lambda+\mu=c(\bar{\gamma}|_{\mathfrak{t}})$, and $\nu=c(\bar{\gamma}|_{\mathfrak{a}})$. Let $\mathfrak{q}=\mathfrak{q}(\lambda+\mu)=\mathfrak{l}\oplus\mathfrak{u}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by*

$$\Delta(\mathfrak{l})=\{\alpha\in\Delta(\mathfrak{g}:\mathfrak{t}):\langle\alpha, \lambda+\mu\rangle=0\}, \qquad \textit{and}$$
$$\Delta(\mathfrak{u})=\{\alpha\in\Delta(\mathfrak{g}:\mathfrak{t}):\langle\alpha, \lambda+\mu\rangle>0\}.$$

Then L is quasisplit, and any minimal K-type of π has a highest weight of the form

$$\lambda+\mu+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L, \tag{3.2.8}$$

where δ_L is a fine weight of L.

Proof. The fact that L is quasisplit is easy to check by writing down $\lambda+\mu$ as an element of $i\mathfrak{t}_0^*$. To finish the proof it is sufficient to show that every minimal K-type of $I(\gamma_d)$ is of the form (3.2.8), where $I(\gamma_d)$ is as in Lemma 3.2.6. Let H_d, λ_d, r, s, etc. be as in the lemma as well. Notice that H_d is a maximally split Cartan subgroup of L, and $\sigma(\Psi_d, \lambda_d)$ is in the discrete series.

Write $(\bar{\gamma}_d)_c = \lambda_d + \mu_d + \nu_d$. Then $\lambda_d + \mu_d = \lambda + \mu$, and $\nu_d = \nu$. If $X(\gamma_d)$ is the $(\mathfrak{g}, K)$-module associated to $I(\gamma_d)$, then by Theorem 11.225 of [KV],

$$X(\gamma_d) \cong \mathscr{L}_S(W), \tag{3.2.9}$$

where $S = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$ and W is the $(\mathfrak{l}, L \cap K)$-module associated to the standard representation $I(\gamma_L)$ of L determined by the following data:

$$H_L = H_d, \qquad \lambda_L + \mu_L = \lambda + \mu - \rho(\mathfrak{u}), \qquad \text{and} \qquad \nu_L = \nu.$$

Since H_L is maximally split for L, $I(\gamma_L)$ is in the principal series of L. Moreover, we have that $\lambda + \mu|_{(\mathfrak{u}(1)^{m_i} \times \mathfrak{u}(1)^{n_i})}$ and $\rho(\mathfrak{u})|_{(\mathfrak{u}(1)^{m_i} \times \mathfrak{u}(1)^{n_i})}$ are both scalars, so that by Lemma 3.2.4 the highest weights of the minimal $(L \cap K)$-types of W are (precisely) those of the form $\lambda + \mu - \rho(\mathfrak{u}) + \delta_L$ for some fine weight δ_L of L. By the proof of Theorem 10.44 of [KV] (this is case (b) in Section XI.12), the minimal K-types of $\mathscr{L}_S(W)$ are precisely those with highest weights of the form

$$\lambda + \mu - \rho(\mathfrak{u}) + \delta_L + 2\rho(\mathfrak{u} \cap \mathfrak{p}) = \lambda + \mu + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L. \quad \blacksquare$$

As a corollary to the proof of Proposition 3.2.7, we have the following

LEMMA 3.2.10. *In the setting of Proposition* 3.2.7, *every genuine K-type with highest weight of the form* (3.2.8) *is a minimal K-type of $I(\gamma_d)$.*

4. THE INDUCTION PRINCIPLE

The main argument in this chapter is due to Steve Kudla [K]. A similar calculation can be found in [AB1].

4.1. *The Schrödinger Model*

Let $(W, \langle\ ,\ \rangle)$ be a symplectic space, $W \cong \mathbb{R}^{2n}$. Fix a symplectic basis with respect to which $\langle\ ,\ \rangle$ is given by $\langle w_1, w_2 \rangle = {}^t w_1 J w_2$, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Write $W = X \oplus Y$ for the corresponding complete polarization of W. Let $Sp = Sp(W) = \{g \in GL(W) : {}^t g J g = J\}$. Denote an element $g \in Sp$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a \in Hom(X, X), \quad b \in Hom(Y, X),$$

$$c \in Hom(X, Y), \quad \text{and} \quad d \in Hom(Y, Y). \qquad (4.1.1)$$

Identify $Y \cong X^*$ and $X \cong Y^*$ via the nondegenerate pairing $\langle\ ,\ \rangle$: $X \times Y \to \mathbb{R}$.

Let P be the Siegel parabolic

$$P = Stab_{Sp}(X) = \left\{ \begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in GL(X), a\,{}^t b = b\,{}^t a \right\}.$$

Then $P = MN = NM$, where

$$M = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in GL(X) \right\} \cong GL(X), \qquad \text{and}$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b = {}^t b \right\}.$$

Fix a unitary additive character ψ of $\mathbb{R}$. We want to realize ω_ψ on $\mathcal{S}(Y)$, the space of Schwartz functions on Y, as in [R].

Let $\widetilde{Sp} = \widetilde{Sp}(W)$ be the two-fold cover of Sp defined by the normalized cocycle $\tilde{c}$ of Section 5 in [R], and $\omega = \omega_\psi$ the corresponding representation of $\widetilde{Sp}$ realized on $\mathcal{S}(Y)$. Let $\widetilde{M}$ and $\widetilde{P}$ be the inverse images of M and P in $\widetilde{Sp}$ respectively. Notice that $\widetilde{P} \cong \widetilde{M}N$. On $\widetilde{P}$, ω is given by

$$\omega(p, \varepsilon)\, \varphi(y) = |det(a)|^{1/2}\, \gamma_{\mathbb{R}}(det(a), \tfrac{1}{2}\psi)^{-1}\, \psi(\tfrac{1}{2}\langle b\,{}^t a y, y\rangle)\, \varphi({}^t a y)\, \varepsilon, \qquad (4.1.2)$$

where $\varphi \in \mathcal{S}(Y)$, $p = \left(\begin{smallmatrix} a & b \\ 0 & {}^t a^{-1} \end{smallmatrix}\right)$, and if ψ is given by $\psi(r) = e^{i\xi r}$, then

$$\gamma_{\mathbb{R}}(det(a), \tfrac{1}{2}\psi) = \begin{cases} 1 & \text{if } \ det(a) > 0, \\ -i sgn(\xi) & \text{otherwise.} \end{cases} \qquad (4.1.3)$$

On $\widetilde{P}$, $\tilde{c}$ is given by $\tilde{c}(p_1, p_2) = (det(a_1), det(a_2))_{\mathbb{R}}$, where

$$p_i = \begin{pmatrix} a_i & b_i \\ 0 & {}^t a_i^{-1} \end{pmatrix}$$

and $(\ ,\ )_{\mathbb{R}}$ is the Hilbert symbol of $\mathbb{R}$.

4.2. *The Mixed Model* [H2]

Suppose W_1 and W_2 are both symplectic spaces, and let Sp_1 and Sp_2 be the corresponding symplectic groups, with normalized covers $\widetilde{Sp}_i$ and oscillator representations $\omega_i = (\omega_i)_\psi$. Let $W = W_1 \oplus W_2$, and $Sp = Sp(W)$, and let $\omega = \omega_\psi$ be the oscillator representation of $\widetilde{Sp}$. Then there is a natural map $\alpha\colon Sp_1 \times Sp_2 \to Sp$. By Prop. 3.7 and Cor. 5.6 of [R], α lifts to the covers, with

$$\tilde{\alpha}((g_1, \varepsilon_1), (g_2, \varepsilon_2)) = (\alpha(g_1, g_2), \varepsilon_1 \varepsilon_2 (x(g_1), x(g_2))_{\mathbb{R}}), \tag{4.2.1}$$

and we have that

$$\omega \circ \tilde{\alpha} \cong \omega_1 \otimes \omega_2. \tag{4.2.2}$$

Now let $W = X \oplus W^0 \oplus Y$ be a symplectic space, with X and Y totally isotropic subspaces which are dual to each other. (This corresponds to taking $W_1 = W^0$ and $W_2 = X \oplus Y$ in the above formulas.) Let $W^0 = W^0 \oplus Y^0$ be a complete polarization of W^0 (as in 4.1). Then we have as before $Y \cong X^*$, $Y^0 \cong (X^0)^*$, and $W = (X \oplus X^0) \oplus (Y \oplus Y^0)$ is a complete polarization of W. This defines covers and oscillator representations ω_0 and ω of $\widetilde{Sp}(W^0)$ and $\widetilde{Sp}(W)$, realized on $\mathscr{S}(Y^0)$ and $\mathscr{S}(Y \oplus Y^0) \cong \mathscr{S}(Y) \oplus \mathscr{S}(Y^0)$ respectively. Let $P = MN$ be the stabilizer of X in $Sp(W)$. Then $M \cong GL(X) \times Sp(W^0)$, with the map

$$\beta\colon GL(X) \times Sp(W^0) \to Sp(W)$$

defined by $\beta(g, h)(x, w, y) = (gx, hw, {}^tg^{-1}y)$ for any $g \in GL(X)$, $h \in Sp(W^0)$, $x \in X$, $w \in W^0$, and $y \in Y$. Notice that $GL(X) \hookrightarrow M_{X \oplus Y} \subset Sp(X \oplus Y)$, where $M_{X \oplus Y}$ is the Levi factor of the Siegel parabolic of $Sp(X \oplus Y)$.

Recall that $\mathscr{S}(Y) \otimes \mathscr{S}(Y^0)$ is generated by products $\varphi\phi$ with $\varphi \in \mathscr{S}(Y)$ and $\phi \in \mathscr{S}(Y^0)$. Now (4.1.2), together with Proposition 3.7 and the proof of Corollary 5.6 of [R], yield that the action of ω restricted to the inverse image $\tilde{M}$ of M in $\widetilde{Sp}(W)$, is given by

$$\begin{aligned} &\omega(\beta(g, h), \varepsilon)\, \varphi(y)\, \phi(y_0) \\ &\quad = |det(g)|^{1/2}\, \gamma_{\mathbb{R}}(det(g), \tfrac{1}{2}\psi)^{-1}\, (det(g), x(h))_{\mathbb{R}}\, \varphi({}^tgy)\, \omega_0(h, \varepsilon)\, \phi(y_0). \end{aligned} \tag{4.2.3}$$

Here the map x and the multiplier m are those of [R] and should in each case be understood in the context of the appropriate symplectic group.

4.3. *The Dual Pair* $U(V)\times U(W)$

Let V and W be complex spaces endowed with a hermitian and a skew-hermitian form respectively, and let $U(V)$ and $U(W)$ be the corresponding isometry groups. Suppose

$$V=V_+\oplus V^0\oplus V_- \qquad \text{and} \qquad W=W_+\oplus W^0\oplus W_-, \tag{4.3.1}$$

where V_+ and V_-, as well as W_+ and W_-, are totally isotropic subspaces which are dual to each other. Let $P_V=P_V(V_+)$ be the stabilizer of V_+ in $U(V)$, and similarly $P_W=P_W(W_+)$. Then

$$\begin{aligned} P_V&=M_VN_V \qquad \text{with} \quad M_V\cong GL(V_+)\times U(V^0), \qquad \text{and}\\ P_W&=M_WN_W \qquad \text{with} \quad M_W\cong GL(W_+)\times U(W^0). \end{aligned} \tag{4.3.2}$$

Here $GL(V_+)\cong GL(t_V,\mathbb{C})$ and $GL(W_+)\cong GL(t_W,\mathbb{C})$, where t_V and t_W are the dimensions of V_+ and W_+ over $\mathbb{C}$.

Now let $\mathbb{W}=V\otimes W$. $\mathbb{W}$ is a skew-hermitian space, and when regarded as a real vector space, a symplectic space over $\mathbb{R}$. Let $Sp=Sp(\mathbb{W})$ be the corresponding symplectic group. If $V^0\otimes W^0=X^0\oplus Y^0$ is a complete polarization of the real symplectic space $V^0\otimes W^0$, then $\mathbb{W}=\mathbb{X}\oplus\mathbb{Y}$ is a complete polarization of $\mathbb{W}$, where

$$\begin{aligned} \mathbb{X}&=(V\otimes W_+)\oplus(V_+\otimes W^0)\oplus X^0, \qquad \text{and}\\ \mathbb{Y}&=(V\otimes W_-)\oplus(V_-\otimes W^0)\oplus Y^0. \end{aligned} \tag{4.3.3}$$

We realize the oscillator representation ω of $\widetilde{Sp}$ on $\mathscr{S}(\mathbb{Y})$. Let

$$W_M=(V_-\otimes W_+)\oplus(V_+\otimes W_-)\oplus(V^0\otimes W^0). \tag{4.3.4}$$

Then (M_V, M_W) is a dual pair in $Sp(W_M)$, and the oscillator representation ω_M of $\widetilde{Sp}(W_M)$ may be realized on $\mathscr{S}(\mathbb{Y}_M)$, where $\mathbb{Y}_M=(V_+\otimes W_-)\oplus Y^0$. Let

$$\rho\colon \mathscr{S}(\mathbb{Y})\to\mathscr{S}(\mathbb{Y}_M) \tag{4.3.5}$$

be the restriction map. Notice that ρ is surjective.

Now $GL(V_+)\times GL(W_+)$ is a dual pair in $Sp((V_+\otimes W_-)\oplus(V_-\otimes W_+))$, and we shall see that the associated covers are trivial. Similarly, $U(V^0)\times U(W^0)$ is a dual pair in $Sp(V^0\otimes W^0)$. Let $\tilde{U}(V^0)$ and $\tilde{U}(W^0)$ be the covers coming from $\widetilde{Sp}(V^0\otimes W^0)$. Then by (4.2.1) there is a map

$$GL(V_+)\times GL(W_+)\times\tilde{U}(V^0)\times\tilde{U}(W^0)\to\widetilde{Sp}(W_M)\to\widetilde{Sp}. \tag{4.3.6}$$

The image of this map in $\widetilde{Sp}$ is $\tilde{M}_V \times \tilde{M}_W$, where the covers of M_V and M_W are those inherited from $\widetilde{Sp}$. The cover of $\widetilde{Sp}$ restricted to N_V and N_W is trivial since N_V and N_W are simply connected. Let $\tilde{P}_V = \tilde{M}_V N_V$ and $\tilde{P}_W = \tilde{M}_W N_W$. We want to show that ρ in (4.3.5) is a $\tilde{P}_V \times \tilde{P}_W$-equivariant map

$$\omega \to \omega_M \otimes \xi \tag{4.3.7}$$

for some character ξ of $\tilde{P}_V \times \tilde{P}_W$, with N_V and N_W acting trivially on the right hand side.

First we consider the actions of N_V and N_W. It is easy to check that both groups imbed into the Siegel parabolic $P_S = P_S(\mathbb{X}) \subseteq Sp$, so that a direct calculation using (4.1.2) yields

LEMMA 4.3.8. *The map ρ in (4.3.5) is an $N_V \times N_W$-map with $N_V \times N_W$ acting trivially on $\mathscr{S}(\mathbb{Y}_M)$.*

4.4. *The Action of $GL(V_+)$ and $GL(W_+)$*

For temporary notation, let

$$\begin{aligned} W_1 &= (V_- \otimes W_+) \oplus (V_+ \otimes W_-) = X_1 \oplus Y_1, \\ W_2 &= V^0 \otimes W^0 = X^0 \oplus Y^0, \end{aligned} \tag{4.4.1}$$

and let ω_1 and ω_2 be the oscillator representations of $\widetilde{Sp}(W_1)$ and $\widetilde{Sp}(W_2)$ respectively. We have that $GL(V_+) \times GL(W_+)$ is a dual pair in $Sp(W_1)$. If $P_1 = M_1 N_1 = P_1(X_1)$ is the Siegel parabolic of $Sp(W_1)$, then $GL(V_+)$ and $GL(W_+)$ imbed into M_1 as follows: For $g \in GL(V_+)$ and $h \in GL(W_+)$,

$$g \mapsto \begin{pmatrix} a(g) & 0 \\ 0 & {}^t a(g)^{-1} \end{pmatrix} \quad \text{and} \quad h \mapsto \begin{pmatrix} a'(h) & 0 \\ 0 & {}^t a'(h)^{-1} \end{pmatrix}, \tag{4.4.2}$$

where

$$\begin{aligned} a(g) \cdot (v_- \otimes w_+) &= ((g^*)^{-1} \cdot v_-) \otimes w_+ \quad \text{and} \\ a'(h) \cdot (v_- \otimes w_+) &= v_- \otimes (h \cdot w_+). \end{aligned} \tag{4.4.3}$$

Here g^* denotes the adjoint of g with respect to the skew-hermitian form on $((V_+ \otimes W_-) \oplus (V_- \otimes W_+))$. The determinants of $a(g)$ and $a'(h)$ as elements of $GL(X_1)$ are then

$$det(a(g)) = |det(g)|^{-2t_W} \quad \text{and} \quad det(a'(h)) = |det(h)|^{2t_V}. \tag{4.4.4}$$

In particular, they are both positive, so that the covers of $GL(V_+)$ and $GL(W_+)$ inside $\widetilde{Sp}(W_1)$ are indeed trivial. With the natural maps of (4.3.6) understood (but not written), and using (4.2.3) as well as the isomorphism $\mathscr{S}(\mathbb{Y}_M) \cong \mathscr{S}(Y_1) \otimes \mathscr{S}(Y^0)$, we have that for $g \in GL(V_+)$, $h \in GL(W_+)$, $u_V \in \tilde{U}(V^0)$, $u_W \in \tilde{U}(W^0)$, $\varphi \in \mathscr{S}(Y_1)$, and $\phi \in \mathscr{S}(Y^0)$,

$$\begin{aligned}\omega_M(g, h, u_V, u_W)\, \varphi(y_1)\, \phi(y))\\ = |det(g)|^{-t_W}\, |det(h)|^{t_V}\, \varphi({}^t a(g)^t\, a'(h)\, y_1)\, \omega_2(u_V, u_W)\, \phi(y_0). \quad (4.4.5)\end{aligned}$$

Since $GL(V_+)$ and $GL(W_+)$ imbed into $GL(\mathbb{X})$, it is now a straightforward calculation to compare the actions of $GL(V_+)$ and $GL(W_+)$ on $\mathscr{S}(\mathbb{Y}_M)$ via ω_M with those obtained by restriction from $\mathscr{S}(\mathbb{Y})$, and we obtain

LEMMA 4.4.6. *Let $m = \dim_{\mathbb{C}}(W^0)$ and $n = \dim_{\mathbb{C}}(V^0)$. Let χ_V and χ_W be the characters of $GL(V_+)$ and $GL(W_+)$ given by $\chi_V(g) = |det(g)|^{m+t_W}$ and $\chi_W(h) = det(h)|^{n+t_V}$. The map ρ is a $GL(V_+) \times GL(W_+)$ map*

$$\omega \to \omega_M \otimes \chi_V \otimes \chi_W, \quad (4.4.7)$$

4.5. *The Action of $\tilde{U}(V^0)$ and $\tilde{U}(W^0)$*

To understand the action of $\tilde{U}(V^0)$ and $\tilde{U}(W^0)$ on $\mathscr{S}(\mathbb{Y})$ and $\mathscr{S}(Y^0)$, write

$$\begin{aligned}\mathbb{W} = (X \oplus X^0) \oplus (Y \oplus Y^0) = \mathbb{X} \oplus \mathbb{Y}\\ \text{with} \quad X = (V \otimes W_+) \oplus (V_+ \otimes W^0)\\ \text{and} \quad Y = (V \otimes W_-) \oplus (V_- \otimes W^0). \quad (4.5.1)\end{aligned}$$

Let $P = P(X) = MN \subset Sp$, with $M \cong GL(X) \times Sp(V^0 \otimes W^0)$. Then both $U(V^0)$ and $U(W^0)$ imbed into M. Using (4.2.3), we can compute the action of $\tilde{U}(V^0) \times \tilde{U}(W^0)$ on $\mathscr{S}(\mathbb{Y})$, and by considering the restriction to $\mathscr{S}(\mathbb{Y}_M)$ and comparing with (4.4.5), we obtain

LEMMA 4.5.2. *ρ is a $\tilde{U}(V^0) \times \tilde{U}(W^0)$ map $\omega \to \omega_M$.*

Putting together Lemmas 4.4.9, 4.4.6, and 4.5.2, we have the following

THEOREM 4.5.3. *Let ξ be the following character of $GL(V_+) \times GL(W_+)$:*

$$\xi((g_V, g_W)) = |det(g_V)|^{m+t_W}\, |det(g_W)|^{n+t_V}.$$

Then the surjective map ρ defined in (4.3.5) *is a $\tilde{P}_V \times \tilde{P}_W$-equivariant map*

$$\omega \to \omega_M \otimes \xi.$$

Recall that $\tilde{P}_V = \tilde{M}_V N_V \subset \tilde{U}(V)$ and $\tilde{P}_W = \tilde{M}_W N_W \subset \tilde{U}(W)$. Let $\mathfrak{n}_V$ and $\mathfrak{n}_W$ be the Lie algebras of N_V and N_W, and let $\rho(\mathfrak{n}_V)$ and $\rho(\mathfrak{n}_W)$ be one half the sums of the roots of $\mathfrak{n}_V$ and $\mathfrak{n}_W$ respectively. Then $\rho(\mathfrak{n}_V)$ and $\rho(\mathfrak{n}_W)$ exponentiate to the characters ξ_V and ξ_W of $\tilde{P}_V$ and $\tilde{P}_W$ given by

$$\begin{aligned} \xi_V(g_V, u_V, n_V) &= |det(g_V)|^{n+t_V}, \qquad \text{and} \\ \xi_W(g_W, u_W, n_W) &= |det(g_W)|^{m+t_W}, \end{aligned} \tag{4.5.4}$$

where $g_V \in GL(V_+)$, $u_V \in \tilde{U}(V^0)$, $n_V \in N_V$, and similarly for g_W, u_W, and n_W. Using Theorem 4.5.3 and Frobenius Reciprocity, we get the following result.

THEOREM 4.5.5. *Suppose* $\pi_V \in \tilde{U}(V^0)\hat{}$, $\pi_W \in \tilde{U}(W^0)\hat{}$, $\sigma_V \in GL(V_+)\hat{}$, *and* $\sigma_W \in GL(W_+)\hat{}$, *and suppose that* $\pi_V \leftrightarrow \pi_W$ *and* $\sigma_V \leftrightarrow \sigma_W$ *in the correspondences for the dual pairs* $U(V^0) \times U(W^0)$ *and* $GL(V_+) \times GL(W_+)$. *Let* χ_V *and* χ_W *be the characters of* $GL(V_+)$ *and* $GL(W_+)$ *given by*

$$\begin{aligned} \chi_V(g_V) &= |det(g_V)|^{m+t_W-n-t_V}, \qquad \textit{and} \\ \chi_W(g_W) &= |det(g_W)|^{n+t_V-m-t_W}. \end{aligned} \tag{4.5.6}$$

Then there is a nonzero $\tilde{U}(V) \times \tilde{U}(W)$*-map* (*of Harish–Chandra modules for* $\tilde{U}(V) \times \tilde{U}(W)$)

$$\omega \to Ind_{\tilde{P}_V}^{\tilde{U}(V)}(\pi_V \otimes \sigma_V \otimes \chi_V) \otimes Ind_{\tilde{P}_W}^{\tilde{U}(W)}(\pi_W \otimes \sigma_W \otimes \chi_W). \tag{4.5.7}$$

4.6. *The Dual Pair* $GL(V_+) \times GL(W_+)$

Theorem 4.6 of [AB1] gives the correspondence for dual pairs of the form $GL(t, \mathbb{C}) \times GL(s, \mathbb{C}) \subseteq Sp(4ts, \mathbb{R})$. Taking into account the particular imbedding we have chosen, we have for the case $t = s$:

PROPOSITION 4.6.1. *For every* $\sigma \in GL(t, \mathbb{C})\hat{}$, σ *occurs in the correspondence, and* $\sigma \leftrightarrow \check{\sigma}$, *where* $\check{\sigma}(g) = \sigma^*(\bar{g})$, *and* $\bar{g}$ *denotes the complex conjugate of* g.

In the case $t = 1$, we have that $GL(1, \mathbb{C}) \cong \mathbb{C}^\times$, and if $\sigma = \sigma_{\nu, k}$ is a character of $\mathbb{C}^\times$, given by $\sigma(re^{i\theta}) = r^\nu e^{ik\theta}$, then

$$\check{\sigma}(re^{i\theta}) = \sigma^*(re^{-i\theta}) = \sigma\left(\frac{1}{r} e^{i\theta}\right) = r^{-\nu} e^{ik\theta} = \sigma_{-\nu, k}(re^{i\theta}). \tag{4.6.2}$$

PROPOSITION 4.6.3. *The bijection of $U(t)$-types in the space of joint harmonics for $GL(t, \mathbb{C}) \times GL(t, \mathbb{C})$ is given by*

$$(a_1, ..., a_k, 0, ..., 0, b_1, ..., b_\ell) \leftrightarrow (a_1, ..., a_k, 0, ..., 0, b_1, ..., b_\ell),$$

and the degree of this K-type is $\sum_{i=1}^k a_i - \sum_{i=1}^\ell b_i$.

Proof. This follows from Proposition 1.4 of [AB1]. ∎

THEOREM 4.6.4. *Let $p+q=r+s$, let k be a positive integer, and suppose that $\pi \leftrightarrow \pi'$ in the correspondence for the dual pair $U(p, q) \times U(r, s)$, with $\pi \in \widetilde{U}(p, q)\widehat{\ }_{genuine}$, $\pi' \in \widetilde{U}(r, s)\widehat{\ }_{genuine}$. Let $\sigma \in GL(k, \mathbb{C})\widehat{\ }$, and let ω be the oscillator representation for the dual pair*

$$\widetilde{U}(p+k, q+k) \times \widetilde{U}(r+k, s+k) = G_1 \times G_2 \subset \widetilde{Sp}(2(p+q+2k)^2, \mathbb{R}).$$

Then there is a $G_1 \times G_2$-map (of Harish–Chandra modules for $G_1 \times G_2$)

$$\phi\colon \omega \to Ind_{P_1}^{G_1}(\pi \otimes \sigma \otimes \mathbb{1}) \otimes Ind_{P_2}^{G_2}(\pi' \otimes \check{\sigma} \otimes \mathbb{1}), \tag{4.6.5}$$

where $P_1 = M_1 N_1$ and $P_2 = M_2 N_2$ are parabolic subgroups of G_1 and G_2 with Levi factors $M_1 \cong \widetilde{U}(p, q) \times GL(k, \mathbb{C})$ and $M_2 \cong \widetilde{U}(r, s) \times GL(k, \mathbb{C})$, and $\check{\sigma} \in GL(k, \mathbb{C})\widehat{\ }$ is obtained from σ as in Proposition 4.6.1.

Proof. This follows from Theorem 4.5.5 and Proposition 4.6.1. ∎

THEOREM 4.6.6. *In the setting of Theorem* 4.6.4., *let K_1 and K_2 be maximal compact subgroups of G_1 and G_2 respectively. Let ω_M be the oscillator representation for the dual pair $M_1 \times M_2$, as in Theorem* 4.5.3.

Suppose μ_1 is a K_1-type for G_1, λ_1 is a $(K_1 \cap M_1)$-type for M_1, and that μ_1 and λ_1 satisfy the following properties:

(1) *λ_1 is of minimal degree in $\pi \otimes \sigma$.*

(2) *μ_1 is of minimal degree and of multiplicity one in $Ind_{P_1}^{G_1}(\pi \otimes \sigma \otimes \mathbb{1})$.*

(3) $\deg(\mu_1) = \deg(\lambda_1)$, *and the restriction of μ_1 to $(K_1 \cap M_1)$ contains λ_1.*

(4) *There exist characters α_1 and α_2 of M_1 and M_2 which are trivial on $(K_1 \cap M_1)$ and $(K_2 \cap M_2)$, and such that $(\pi \otimes \sigma \otimes \alpha_1) \otimes (\pi' \otimes \check{\sigma} \otimes \alpha_2)$ is a quotient of ω_M, and $Ind_{P_1}^{G_1}(\pi \otimes \sigma \otimes \alpha_1 \otimes \mathbb{1})$ is irreducible.*

Let μ_2 be the K_2-type which corresponds to μ_1 in the space of joint harmonics, and let ϕ be the map in (4.6.5). *Then $\mu_1 \otimes \mu_2$ is in the image of ϕ.*

Proof. Analogous to the proof of Proposition 3.25 of [AB1]. ∎

5. SOME K-TYPE CALCULATIONS

5.1. *A Key Lemma*

Let $I(\gamma_d)$ be a genuine standard representation of $\widetilde{U}(p,q)$ induced from discrete series. Recall that by Lemma 3.2.6, every $\pi \in \widetilde{U}(p,q)\widehat{\ }_{genuine}$ is a lowest K-type (LKT) constituent of such a representation. The goal of this section is to show that for a particular choice of r and s with $r+s=p+q$, every LKT of $I(\gamma_d)$ (and hence of each LKT constituent π) occurs in the space of joint harmonics for the dual pair $(U(p,q), U(r,s))$, and moreover is of minimal r,s-degree (see Terminology 1.4.4) in $I(\gamma_d)$, and hence in π.

So let $\gamma_d = (\Psi, \bar{\gamma})$ with $\bar{\gamma}_c = \lambda+\mu+\nu$ and $\mathfrak{q}=\mathfrak{q}(\lambda+\mu)=\mathfrak{l}\oplus\mathfrak{u}$ be defined as in Proposition 3.2.7.

Lemma 5.1.1. *Let σ be the highest weight of a K-type of $I(\gamma_d)$. Then σ is of the form*

$$\sigma = \delta + \sum_{\alpha} n_\alpha \alpha,$$

where the sum runs over roots in $\Delta(\mathfrak{l}:\mathfrak{t}) \cup \Delta(\mathfrak{u}\cap\mathfrak{p})$, $n_\alpha \geqslant 0$ for all α, and δ is the highest weight of a minimal K-type of $I(\gamma_d)$.

Proof. Let $X(\gamma_d) \cong \mathscr{L}_S(W)$ be as in Proposition 3.2.7. By Theorems 0.46 and 11.225 of [KV], $\mathscr{L}_S(W) \equiv \mathscr{R}^S_{\mathfrak{q}}(W)$ in the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules, hence $X(\gamma_d)$ and $\mathscr{R}^S_{\mathfrak{q}}(W)$ have the same K-types. By the generalized Blattner formula (Theorem 6.3.12 of [V2]), the K-types of $\mathscr{R}^S_{\mathfrak{q}}(W)$ have highest weights of the form

$$\sigma = \eta + 2\rho(\mathfrak{u}\cap\mathfrak{p}) + \sum_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p})} n_\alpha \alpha, \tag{5.1.2}$$

where η is the highest weight of an $(L\cap K)$-type of W. By Theorem 10.44 of [Kn], δ is of the form $\delta = \eta' + 2\rho(\mathfrak{u}\cap\mathfrak{p})$ for some minimal $(L\cap K)$-type of W with highest weight η'. The highest weights of any two $(L\cap K)$-types of W differ by a sum of roots in $\Delta(\mathfrak{l})$, so that η is of the form $\eta' + \sum_{\alpha\in\Delta(\mathfrak{l})} n_\alpha \alpha$, and the result follows. ∎

We know that $\lambda+\mu$ is of the form

$$\begin{aligned}\lambda+\mu = (&\underbrace{a_1, ..., a_1}_{k_1}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{l_1}, ..., \underbrace{b_y, ..., b_y}_{l_y};\\ &\underbrace{a_1, ..., a_1}_{m_1}, ..., \underbrace{a_x, ..., a_x}_{m_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{n_1}, ..., \underbrace{b_y, ..., b_y}_{n_y}),\end{aligned} \tag{5.1.3}$$

where $a_1 > \cdots > a_x > 0 > b_1 > \cdots > b_y$, $|k_i - m_i| \leqslant 1$, $|l_i - n_i| \leqslant 1$, and $a_i, b_j \in \frac{1}{2}\mathbb{Z}$ for all i and j. Let $k = \sum_i k_i$, $l = \sum_i l_i$, $m = \sum_i m_i$, and $n = \sum_i n_i$, and fix $r = k + n + w$ and $s = l + m + w$.

PROPOSITION 5.1.4. *Let $\bar{\sigma}$ be a minimal K-type of $I(\gamma_d)$. Then $\bar{\sigma}$ is r, s-harmonic and of minimal r, s-degree in $I(\gamma_d)$.*

Proof. By Proposition 3.2.7 and a straightforward calculation of ρ-shifts, the highest weight σ of $\bar{\sigma}$ is given by

$$\sigma = \left(\frac{r-s}{2}, ..., \frac{r-s}{2}; \frac{s-r}{2}, ..., \frac{s-r}{2}\right)$$
$$+ (\underbrace{A_1, ..., A_1}_{k_1}, ..., \underbrace{A_x, ..., A_x}_{k_x}, \underbrace{0, ..., 0}_{w}, \underbrace{B_1, ..., B_1}_{l_1}, ..., \underbrace{B_y, ..., B_y}_{l_y};$$
$$\underbrace{A'_1, ..., A'_1}_{m_1}, ..., \underbrace{A'_x, ..., A'_x}_{m_x}, \underbrace{0, ..., 0}_{w}, \underbrace{B'_1, ..., B'_1}_{n_1}, ..., \underbrace{B'_y, ..., B'_y}_{n_y}), \tag{5.1.5}$$

where

$$A_i = a_i + \frac{1}{2}\left(\sum_{j=1}^{i-1} (k_j - m_j) + \sum_{j=i+1}^{x} (m_j - k_j) - k + m\right) + \varepsilon_i, \tag{5.1.6a}$$

$$B_i = b_i + \frac{1}{2}\left(\sum_{j=1}^{i-1} (l_j - n_j) + \sum_{j=i+1}^{y} (n_j - l_j) - n + l\right) + \eta_i, \tag{5.1.6b}$$

$$A'_i = a_i + \frac{1}{2}\left(\sum_{j=1}^{i-1} (m_j - k_j) + \sum_{j=i+1}^{x} (k_j - m_j) + k - m\right) - \varepsilon_i, \tag{5.1.6c}$$

$$B'_i = b_i + \frac{1}{2}\left(\sum_{j=1}^{i-1} (n_j - l_j) + \sum_{j=i+1}^{y} (l_j - n_j) + n - l\right) - \eta_i, \tag{5.1.6d}$$

and

$$\begin{aligned} \varepsilon_i &= \begin{cases} \pm\frac{1}{2} & \text{if} \quad k_i = m_i \quad \text{and} \quad a_i \in \mathbb{Z} + \frac{1}{2}, \\ 0 & \text{otherwise}; \end{cases} \\ \eta_i &= \begin{cases} \pm\frac{1}{2} & \text{if} \quad l_i = n_i \quad \text{and} \quad b_i \in \mathbb{Z} + \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned} \tag{5.1.7}$$

Since

$$\sum_{j=1}^{i-1} (k_j - m_j) + \sum_{j=i+1}^{x} (m_j - k_j) - k + m = 2\sum_{j=i+1}^{x} (m_j - k_j) + m_i - k_i,$$

$|m_j - k_j| = 1 \Rightarrow a_j \in \mathbb{Z} + \frac{1}{2}$, and $a_i > a_{i+1} > \cdots > a_x$, we have that

$$2a_i \geqslant \left| \sum_{j=1}^{i-1} (k_j - m_j) + \sum_{j=i+1}^{x} (m_j - k_j) - k + m \right|. \tag{5.1.8}$$

Moreover, $|\varepsilon| \leqslant \frac{1}{2}$, and is 0 if

$$a_i + \frac{1}{2}\left(\sum_{j=1}^{i-1} (k_j - m_j) + \sum_{j=i+1}^{x} (m_j - k_j) - k + m \right) = 0.$$

Consequently, A_i, $A'_i \geqslant 0$ for all i. Similarly, B_i, $B'_i \leqslant 0$ for all i, so that $\bar{\sigma}$ is indeed r, s-harmonic by Lemma 1.4.5(1). Notice also that for all i,

$$A_i + A'_i = 2a_i \qquad \text{and} \qquad B_i + B'_i = 2b_i, \tag{5.1.9}$$

and if $k_i \neq m_i$ ($l_i \neq n_i$) then $\varepsilon_i(\eta_i) = 0$. Lemma 1.4.5(2) implies that

$$deg_{r,s}(\bar{\sigma}) = \sum_{i=1}^{x} (k_i A_i + m_i A'_i) - \sum_{i=1}^{y} (l_i B_i + n_i B'_i), \tag{5.1.10}$$

which is independent of δ_L, so that all minimal K-types of $I(\gamma_d)$ have the same degree.

Now rewrite σ as

$$\sigma = (r, s\text{-shift}) + (x_1, ..., x_{k+w+l};\ y_1, ..., y_{m+w+n}), \tag{5.1.11}$$

where $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k \geqslant 0 = x_{k+1} = \cdots = x_{k+w} \geqslant x_{k+w+1} \geqslant \cdots \geqslant x_{k+w+l}$, and similarly for the y_i's. Then

$$deg_{r,s}(\bar{\sigma}) = \sum_{i=1}^{k} x_i + \sum_{i=1}^{m} y_i - \sum_{i=1}^{l} x_{k+w+i} - \sum_{i=1}^{n} y_{m+w+i}. \tag{5.1.12}$$

Let σ' be the highest weight of any K-type $\bar{\sigma}'$ of $I(\gamma_d)$. By Lemma 5.1.1, σ' is of the form

$$\sigma' = \xi + \sum_{\iota} \alpha_\iota,$$

where ξ is the highest weight of a minimal K-type $\bar{\xi}$ of $I(\gamma_d)$, and each α_ι is of one of the following forms:

$$\begin{array}{ll} e_i - e_j, & \text{where } i \leqslant k \Leftrightarrow j \leqslant k \text{ and } i > k+w \Leftrightarrow j > k+w, \\ f_i - f_j, & \text{where } i \leqslant m \Leftrightarrow j \leqslant m \text{ and } i > m+w \Leftrightarrow j > m+w, \\ e_i - f_j, & \text{where } i > k+w \Rightarrow j > m+w \text{ and } j \leqslant m \Rightarrow i \leqslant k, \\ f_i - e_j, & \text{where } i > m+w \Rightarrow j > k+w \text{ and } j \leqslant k \Rightarrow i \leqslant m. \end{array} \tag{5.1.13}$$

By a simple induction argument on the number of roots it follows that if

$$\sum_{\iota} \alpha_{\iota} = (r_1, ..., r_{k+w+l}; s_1, ..., s_{m+w+n}),$$

then

$$\sum_{i=1}^{k} r_i + \sum_{i=1}^{m} s_i \geqslant \sum_{i=1}^{l} r_{k+w+i} + \sum_{i+1}^{n} s_{m+w+i}. \tag{5.1.14}$$

We may assume that $\bar{\xi} = \bar{\sigma}$. The r, s-degree of $\bar{\sigma}'$ is then

$$\begin{aligned} deg_{r,s}(\bar{\sigma}') &= \sum_{i=1}^{k} |x_i + r_i| + \sum_{i=1}^{w} |r_{k+i}| + \sum_{i=1}^{l} |x_{k+w+i} + r_{k+w+i}| \\ &\quad + \sum_{i=1}^{m} |y_i + s_i| + \sum_{i=1}^{w} |s_{m+i}| + \sum_{i=1}^{n} |y_{m+w+i} + s_{m+w+i}| \\ &\geqslant \sum_{i=1}^{k} (x_i + r_i) + \sum_{i=1}^{l} (-x_{k+w+i} - r_{k+w+i}) + \sum_{i=1}^{m} (y_i + s_i) \\ &\quad + \sum_{i=1}^{n} (-y_{m+w+i} - s_{m+w+i}) \\ &= deg_{r,s}(\bar{\sigma}) + \sum_{i=1}^{k} r_i - \sum_{i-1}^{l} r_{k+w+i} + \sum_{i=1}^{m} s_i - \sum_{i-1}^{n} s_{m+w+i} \\ &\geqslant deg_{r,s}(\bar{\sigma}) \qquad \text{by (5.1.14),} \end{aligned}$$

and the proposition is proved. ∎

5.2. *Limits of Discrete Series*

In the previous section we showed that for any $\pi \in \widetilde{U}(p, q)\widehat{\ }_{genuine}$, the lowest K-types of π are of minimal degree and harmonic for a particular choice of r and s. In the case $\pi = \pi(\lambda, \Psi)$ a limit of discrete series representation, we show that the unique minimal K-type of π corresponds in $\mathscr{H}$ to the minimal K-type of a particular limit of discrete series representation π' of $\widetilde{U}(r, s)$.

DEFINITION 5.2.1. (1) For $\zeta \in \mathbb{Z}$ and $\xi \in \mathbb{C}$, define the character $\chi_{\zeta,\xi}$ of $\mathbb{C}^{\times}$ by $\chi_{\zeta,\xi}(\rho e^{i\theta}) = \rho^{\xi} e^{i\zeta\theta}$.

(2) For t a positive integer, $\zeta = (\zeta_1, ..., \zeta_t) \in \mathbb{Z}^t$, and $\xi = (\xi_1, ..., \xi_t) \in \mathbb{C}^t$, define the character $\chi(\zeta, \xi)$ of $(\mathbb{C}^{\times})^t$ by $\chi(\zeta, \xi) = \bigotimes_{i=1}^{t} \chi_{\zeta_i, \xi_i}$.

DEFINITION 5.2.2. Let (λ, Ψ) be the parameters of a genuine limit of discrete series representation of $\tilde{U}(p, q)$. If

$$\begin{aligned}\lambda = (&\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{b_1, ..., b_1}_{l_1}, ..., \underbrace{b_y, ..., b_y}_{l_y};\\ &\underbrace{a_1, ..., a_1}_{m_1}, ..., \underbrace{a_x, ..., a_x}_{m_x}, \underbrace{b_1, ..., b_1}_{n_1}, ..., \underbrace{b_y, ..., b_y}_{n_y}),\end{aligned} \tag{5.2.3}$$

with $a_1 > a_2 > \cdots > a_x > 0 > b_1 > \cdots > b_y$, let $k = \sum_{i=1}^{x} k_i$, $l = \sum_{i=1}^{y} l_i$, $m = \sum_{i=1}^{x} m_i$ and $n = \sum_{i=1}^{y} n_i$.

Define $r(\lambda) = k + n$ and $s(\lambda) = l + m$. Moreover, define $(\Gamma\lambda, \Gamma\Psi)$ to be the parameters of the limit of discrete series representation of $\tilde{U}(r(\lambda), s(\lambda))$ given as

$$\begin{aligned}\Gamma\lambda = (&\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{b_1, ..., b_1}_{n_1}, ..., \underbrace{b_y, ..., b_y}_{n_y};\\ &\underbrace{a_1, ..., a_1}_{m_1}, ..., \underbrace{a_x, ..., a_x}_{m_x}, \underbrace{b_1, ..., b_1}_{l_1}, ..., \underbrace{b_y, ..., b_y}_{l_y}),\end{aligned} \tag{5.2.4a}$$

and $\Gamma\Psi$ is the following system of positive roots:

$$e_i - e_j \in \Gamma\Psi \Leftrightarrow i < j,$$

$$f_i - f_j \in \Gamma\Psi \Leftrightarrow i < j, \quad \text{and}$$

$$e_i - f_j \in \Gamma\Psi \Leftrightarrow \begin{cases} i \leqslant k, \quad j \leqslant m, \quad \text{and} \quad e_i - f_j \in \Psi, \quad \text{or} \\ i \leqslant k \quad \text{and} \quad j > m, \quad \text{or} \\ i > k, \quad j > m, \quad \text{and} \quad f_{i-k+m} - e_{j-m+k} \in \Psi. \end{cases} \tag{5.2.4b}$$

LEMMA 5.2.5. *Let $\pi = \pi(\lambda, \Psi)$ be a genuine limit of discrete series representation of $\tilde{U}(p, q)$. Let $r = r(\lambda)$, $s = s(\lambda)$, $\lambda' = \Gamma\lambda$, $\Psi' = \Gamma\Psi$, and let π' be the limit of discrete series representation of $\tilde{U}(r, s)$ with parameters (λ', Ψ'). Then the lowest K-types of π and π' correspond in the space of joint harmonics for the dual pair $(U(p, q), U(r, s))$.*

Proof. This follows from a straightforward calculation using Lemma 1.4.5(1). ∎

Because of Lemma 3.2.6 we know that the limit of discrete series representation $\pi = \pi(\lambda, \Psi)$ may be realized as a direct summand of a representation $I(\gamma_d)$ induced from a discrete series $\pi_d \otimes \chi$ on the Levi factor $\tilde{U}(p-t, q-t) \times (\mathbb{C}^\times)^t$ of some cuspidal parabolic subgroup of $\tilde{U}(p, q)$.

By Induction by Stages, there is a unitary irreducible representation ρ of $GL(t, \mathbb{C})$ such that π occurs as a summand of $Ind_{MN}^{\tilde{U}(p,q)}(\pi_d \otimes \rho \otimes \mathbb{1})$, where $M \cong \tilde{U}(p-t, q-t) \times GL(t, \mathbb{C})$. To describe ρ, we need the following

LEMMA 5.2.6. *Let $P_t = M_t N_t$ be a parabolic subgroup of $GL(t, \mathbb{C})$ with Levi factor $M_t \cong (\mathbb{C}^\times)^t$. Let $\chi = \chi(\zeta, \xi)$ be a character of $(\mathbb{C}^\times)^t$ such that $Re\{\langle \xi, \alpha \rangle\} \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{n}_t)$. Then $Ind_{P_t}^{GL(t,\mathbb{C})}(\chi \otimes \mathbb{1})$ has a unique irreducible quotient $\rho_{\zeta,\xi}$. Moreover, $\rho_{\zeta,\xi}$ has a unique lowest K-type $\bar{\zeta}$ of multiplicity one and with highest weight $\tilde{\zeta}$, where $\tilde{\zeta}$ is the unique dominant weight conjugate to $(\zeta_1, ...\zeta_t)$ by the Weyl group of $U(t)$. If χ is unitary then $Ind_{P_t}^{GL(t,\mathbb{C})}(\chi \otimes \mathbb{1})$ is irreducible.*

Proof. This is in [D] and [V4]. ∎

If $\pi = \pi(\lambda, \Psi)$ is a limit of discrete series representation with λ given by (5.2.3), let

$$t = \sum_{i=1}^{x} \min\{k_i, m_i\} + \sum_{i=1}^{y} \min\{l_i, n_i\}. \tag{5.2.7}$$

Let λ_d be the Harish–Chandra parameter of the discrete series representation π_d of $\tilde{U}(p-t, q-t)$ obtained from λ as in Lemma 3.2.6, and let

$$\eta = 2(\underbrace{a_1, ..., a_1}_{\min\{k_1, m_1\}}, ..., \underbrace{a_x, ..., a_x}_{\min\{k_x, m_x\}}, \underbrace{b_1, ..., b_1}_{\min\{l_1, n_1\}}, ..., \underbrace{b_y, ..., b_y}_{\min\{l_y, n_y\}}) \in \mathbb{Z}^t.$$

Then Lemma 5.2.6 implies that π is a summand of $Ind_{MN}^{\tilde{U}(p,q)}(\pi_d \otimes \rho_{\eta,0} \otimes \mathbb{1})$ (with M as defined before the lemma). If K is the maximal compact subgroup of $\tilde{U}(p, q)$ and $\bar{\eta}$ the $U(t)$-type with highest weight η, we have the following relationship between the lowest K-type $\bar{\Lambda}$ of π and the lowest $(K \cap M)$-type $\bar{\Lambda}_d \otimes \bar{\eta}$ of $\pi_d \otimes \rho_{\eta,0}$:

LEMMA 5.2.8. *Let r and s be as in Lemma 5.2.5. Then*

$$\bar{\Lambda}|_{K \cap M} \supset \bar{\Lambda}_d \otimes \bar{\eta} \qquad \textit{and} \tag{5.2.9}$$

$$deg_{r,s}(\bar{\Lambda}) = deg_{r-t,s-t}(\bar{\Lambda}_d) + deg(\bar{\eta}), \tag{5.2.10}$$

where (by Proposition 4.6.3) $\deg(\bar{\eta}) = \sum_{i=1}^{x} 2a_i \min\{k_i, m_i\} - \sum_{i=1}^{y} 2b_i \times \min\{l_i, n_i\}$ *is the degree of $\bar{\eta}$ for the dual pair $(GL(t, \mathbb{C}), GL(t, \mathbb{C}))$.*

Proof. To prove (5.2.9), it suffices to show that

$$\Lambda|_{(\mathfrak{u}(1)^{p-t} \oplus \mathfrak{u}(1)^{q-t} \oplus \mathfrak{u}(1)^t)} = \Lambda_d + \eta, \tag{5.2.11}$$

where Λ_d is the highest weight of $\bar{\Lambda}_d$, and $\mathfrak{u}(1)^t$ is embedded diagonally in $\mathfrak{u}(1)^t \oplus \mathfrak{u}(1)^t \subset \mathfrak{u}(t) \oplus \mathfrak{u}(t)$. By the proof of Proposition 5.1.4 (and with notation established there),

$$\begin{aligned}\Lambda = {} & \left(\frac{r-s}{2}, ..., \frac{r-s}{2}; \frac{s-r}{2}, ..., \frac{s-r}{2}\right) \\ & + (\underbrace{A_1, ..., A_1}_{k_1}, ..., \underbrace{A_x, ..., A_x}_{k_x}, \underbrace{B_1, ..., B_1}_{l_1}, ..., \underbrace{B_y, ..., B_y}_{l_y}; \\ & \underbrace{A'_1, ..., A'_1}_{m_1}, ..., \underbrace{A'_x, ..., A'_x}_{m_x}, \underbrace{B'_1, ..., B'_1}_{n_1}, ..., \underbrace{B'_y, ..., B'_y}_{n_y}),\end{aligned} \tag{5.2.12}$$

where the A_i, B_i, A'_i and B'_i are as in 5.1.6. The fact that $\Lambda|_{\mathfrak{u}(1)^t} = \eta$ now follows from observation 5.1.9.

Suppose λ_d is given by

$$\lambda_d = (a'_1, a'_2, ..., a'_u, b'_1, ..., b'_v; c'_1, ..., c'_h, d'_1, ..., d'_w),$$

with $a'_u > 0 > b'_1$ and $c'_h > 0 > d'_w$. Define

$$\begin{aligned} &\text{for} \quad 1 \leqslant i \leqslant u, \quad \alpha_i = \#\{j \leqslant h : a'_i > c'_j\}, \\ &\text{for} \quad 1 \leqslant i \leqslant v, \quad \beta_i = \#\{j \leqslant w : b'_i > d'_j\}, \\ &\text{for} \quad 1 \leqslant i \leqslant h, \quad \gamma_i = \#\{j \leqslant u : c'_i > a'_j\}, \\ &\text{for} \quad 1 \leqslant i \leqslant w, \quad \delta_i = \#\{j \leqslant v : d'_i > b'_j\}. \end{aligned} \tag{5.2.13}$$

Notice that

$$\begin{aligned} &\text{if} \quad a'_i = a_\zeta \quad \text{then} \quad \alpha_i = \sum_{j=\zeta+1}^{x} (m_j - k_j) + (u-i), \\ &\text{if} \quad b'_i = b_\zeta \quad \text{then} \quad \beta_i = \sum_{j=\zeta+1}^{y} (n_j - l_j) + (v-i), \\ &\text{if} \quad c'_i = a_\zeta \quad \text{then} \quad \gamma_i = \sum_{j=\zeta+1}^{x} (k_j - m_j) + (h-i), \\ &\text{if} \quad d'_i = b_\zeta \quad \text{then} \quad \delta_i = \sum_{j=\zeta+1}^{y} (l_j - n_j) + (w-i). \end{aligned} \tag{5.2.14}$$

The weight Λ_d is given by $\Lambda_d = \lambda_d + \rho_n - \rho_c$, where ρ_n and ρ_c are the shifts for the discrete series, and this comes out to be

$$\Lambda_d = (\tilde{a}_1, ..., \tilde{a}_u, \tilde{b}_1, ..., \tilde{b}_v; \tilde{c}_1, ..., \tilde{c}_h, \tilde{d}_1, ..., \tilde{d}_w), \tag{5.2.15}$$

where

$$\begin{aligned}
\tilde{a}_i &= a'_i + \frac{w-h}{2} - \frac{u+v+1}{2} + i + \alpha_i,\\
\tilde{b}_i &= b'_i + \frac{-h-w}{2} - \frac{-u+v+1}{2} + i + \beta_i,\\
\tilde{c}_i &= c'_i + \frac{-u+v}{2} - \frac{h+w+1}{2} + i + \gamma_i,\\
\tilde{d}_i &= d'_i + \frac{-u-v}{2} - \frac{-h+w+1}{2} + i + \delta_i.
\end{aligned} \tag{5.2.16}$$

Notice that

$$k-m=u-h \qquad \text{and} \qquad l-n=v-w. \tag{5.2.17}$$

Now if $a'_i = a_\zeta$ then

$$\begin{aligned}
\tilde{a}_i &= a_\zeta + \frac{w-h}{2} - \frac{u+v+1}{2} + i + \sum_{j=\zeta+1}^{x} (m_j - k_j) + u - i\\
&= a_\zeta + \frac{n-l}{2} + \frac{k-m}{2} + \frac{1}{2} + \sum_{j=\zeta+1}^{x} (m_j - k_j)\\
&= a_\zeta + \frac{1}{2}\left(n - l + \sum_{j=1}^{\zeta} (k_j - m_j) + \sum_{j=\zeta+1}^{x} (m_j - k_j) + 1\right)\\
&= A_\zeta + \frac{k+n-l-m}{2} \qquad \text{since} \quad k_\zeta - m_\zeta = 1 \quad \text{and} \quad \varepsilon_\zeta = 0\\
&= A_\zeta + \frac{r-s}{2}.
\end{aligned} \tag{5.2.18}$$

Similarly we get that if $b'_i = b_\zeta$ then $\tilde{b}_i = B_\zeta + (r-s)/2$; if $c'_i = a_\zeta$ then $\tilde{c}_i = A'_\zeta - (r-s)/2$; if $d'_i = b_\zeta$ then $\tilde{d}_i = B'_\zeta - (r-s)/2$. Consequently,

$$\Lambda|_{(\mathfrak{u}(1)^{p-t} \oplus \mathfrak{u}(1)^{q-t})} = \Lambda_d, \tag{5.2.19}$$

and (5.2.9) is proved.

Recall from (5.1.10) that

$$deg_{r,s}(\bar{\Lambda}) = \sum_{i=1}^{x} (k_i A_i + m_i A'_i) - \sum_{i=1}^{y} (l_i B_i + n_i B'_i).$$

Now

$$deg_{r-t,\,s-t}(\bar{\Lambda}_d)=\sum_{i=1}^{u}\left(\tilde{a}_i-\frac{r-s}{2}\right)-\sum_{i=1}^{v}\left(\tilde{b}_i-\frac{r-s}{2}\right)$$
$$+\sum_{i=1}^{h}\left(\tilde{c}_i-\frac{s-r}{2}\right)-\sum_{i=1}^{t}\left(\tilde{d}_i-\frac{s-r}{2}\right).$$

Using (5.2.19), we have

$$\begin{aligned} &deg_{r,s}(\bar{\Lambda})-deg_{r-d,\,s-d}(\bar{\Lambda}_d)\\ &\quad=\sum_{i=1}^{x}\min\{k_i,m_i\}(A_i+A'_i)-\sum_{i=1}^{y}\min\{l_i,n_i\}(B_i+B'_i)\\ &\quad=\sum_{i=1}^{x}2a_i\min\{k_i,m_i\}-\sum_{i=1}^{y}2b_i\min\{l_i,n_i\}\qquad\text{by (5.1.9)},\end{aligned}$$

and the lemma is proved. ∎

5.3. *General Representations*

We would like to state and prove analogues of Lemmas 5.2.5 and 5.2.8 for irreducible representations π which are not limits of discrete series. In particular, after picking r and s as in Proposition 5.1.4, we shall choose a standard representation of $\tilde{U}(r,s)$ with a unique LKT constituent π' which will be a candidate for $\theta_{r,s}(\pi)$. Unfortunately, Proposition 3.2.7 only tells us that the minimal K-types of π and π' belong to specified finite sets of K-types, so that we cannot be as precise as in Lemma 5.2.5. However, it will be sufficient to establish that the minimal K-types of the associated standard representations induced from discrete series correspond.

LEMMA 5.3.1. *Let* (γ_d), λ, μ, ν, r, s, *and* $\bar{\sigma}$ *be as in Proposition* 5.1.4. *Let* $I(\gamma')$ *be the standard representation of* $\tilde{U}(r,s)$ *given by* $\gamma'=(\Psi',\bar{\gamma}')$ *with* $\bar{\gamma}'_c=\lambda'+\mu+\nu$, *where* $\Psi'=\Gamma\Psi$ *and* $\lambda'=\Gamma\lambda$. *Let* $\bar{\sigma}'$ *be the K-type for* $\tilde{U}(r,s)$ *corresponding to* $\bar{\sigma}$ *in the space of joint harmonics. Then* $\bar{\sigma}'$ *is a minimal* K-*type of* $I(\gamma')$.

Proof. By Lemma 3.2.10, it is sufficient to show that the highest weight of $\bar{\sigma}'$ is of the form (3.2.8). This is a straightforward calculation, using (5.1.5) and Lemma 1.4.5. We omit the details. ∎

In order to state a suitable analogue of Lemma 5.2.8, realize π as the *unique* LKT constituent $\bar{I}(\gamma)$ of a standard representation $I(\gamma)$. Recall from Section 3 that $\gamma=(\Psi,\bar{\gamma})$ and $\bar{\gamma}=\lambda+d_\chi$, and there exists t such that

$\pi_t = \pi_t(\lambda, \Psi)$ is a limit of discrete series representation of $\tilde{U}(p-t, q-t)$, with a unique minimal K-type $\bar{\Lambda}$. If $\chi = \chi(\zeta, \xi)$ with $\zeta = (\zeta_1, ..., \zeta_t)$, we may assume that $\zeta_1 \geqslant \cdots \geqslant \zeta_t$. Consider the induced representation

$$I = Ind_{MN}^{\tilde{U}(p,q)}(\pi_t \otimes \rho_{\zeta,\xi} \otimes \mathbb{1}), \tag{5.3.2}$$

where $M \cong \tilde{U}(p-t, q-t) \times GL(t, \mathbb{C})$. We shall see in the next section (Lemma 5.4.1) that π is also the unique lowest K-type constituent of I. As before, let K be the maximal compact subgroup of $\tilde{U}(p, q)$, and let $r = r(\lambda) + t$ and $s = s(\lambda) + t$.

LEMMA 5.3.3. *Let $\bar{\sigma}$ be a minimal K-type of I, and let $\bar{\zeta}$ be the $U(t)$-type with highest weight ζ. Then*

$$\bar{\sigma}|_{K\cap M} \supset \bar{\Lambda} \otimes \bar{\zeta} \qquad and \tag{5.3.4}$$

$$deg_{r,s}\bar{\sigma} = deg_{r-t,s-t}(\bar{\Lambda}) + deg(\bar{\zeta}), \tag{5.3.5}$$

where $\deg(\bar{\zeta}) = \sum_{i=1}^t |\zeta_i|$ *is the degree of the $U(t)$-type $\bar{\zeta}$ for the dual pair $(GL(t, \mathbb{C}), GL(t, \mathbb{C}))$.*

Proof. Let Λ and σ be the highest weights of $\bar{\Lambda}$ and $\bar{\sigma}$ respectively. As for Lemma 5.2.8, we will show that

$$\sigma|_{(\mathfrak{u}(1)^{p-t} \oplus \mathfrak{u}(1)^{q-t} \oplus \mathfrak{u}(1)^t)} = \Lambda + \zeta, \tag{5.3.6}$$

where $\mathfrak{u}(1)^t$ is embedded diagonally in $\mathfrak{u}(1)^t \oplus \mathfrak{u}(1)^t \subset \mathfrak{u}(t) \oplus \mathfrak{u}(t)$. This will imply (5.3.4).

Suppose λ is given by (5.2.3). In the notation of Proposition 3.2.7, $\lambda + \mu$ is then

$$\lambda + \mu = \lambda + (\tfrac{1}{2}\zeta_1, ..., \tfrac{1}{2}\zeta_t; \tfrac{1}{2}\zeta_1, ..., \tfrac{1}{2}\zeta_t), \tag{5.3.7}$$

which can be written in the form (if necessary, by inserting a_i's with $k_i = m_i = 0$ and b_i's with $l_i = n_i = 0$ in formula (5.2.3))

$$\begin{aligned} \lambda + \mu = (&\underbrace{a_1, ..., a_1}_{k_1 + c_1}, ..., \underbrace{a_x, ..., a_x}_{k_x + c_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{l_1 + d_1}, ..., \underbrace{b_y, ..., b_y}_{l_y + d_y}; \\ &\underbrace{a_1, ..., a_1}_{m_1 + c_1}, ..., \underbrace{a_x, ..., a_x}_{m_x + c_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{n_1 + d_1}, ..., \underbrace{b_y, ..., b_y}_{n_y + d_y}), \end{aligned} \tag{5.3.8}$$

with $\sum_{i=1}^x c_i + \sum_{i=1}^y d_i + w = t$.

Let $c = \sum_{i=1}^x c_i$ and $d = \sum_{i=1}^y d_i$ Using the computations in the proof of Proposition 5.1.4, we get that σ is of the form

$$\begin{aligned}\sigma &= \lambda+\mu+\rho(\mathfrak{u}\cap\mathfrak{p})-\rho(\mathfrak{u}\cap\mathfrak{k})+\delta_L \qquad \text{for some fine weight } \delta_L \text{ of } L,\\ &= \left(\frac{r-s}{2}, ..., \frac{r-s}{2}; \frac{s-r}{2}, ..., \frac{s-r}{2}\right)\\ &\quad +(\underbrace{A_1, ..., A_1}_{k_1+c_1}, ..., \underbrace{A_x, ..., A_x}_{k_x+c_x}, \underbrace{0, ..., 0}_{w}, \underbrace{B_1, ..., B_1}_{l_1+d_1}, ..., \underbrace{B_y, ..., B_y}_{l_y+d_y};\\ &\quad \underbrace{A'_1, ..., A'_1}_{m_1+c_1}, ..., \underbrace{A'_x, ..., A'_x}_{m_x+c_x}, \underbrace{0, ..., 0}_{w}, \underbrace{B'_1, ..., B'_1}_{n_1+d_1}, ..., \underbrace{B'_y, ..., B'_y}_{n_y+d_y}),\end{aligned}$$

where the values for the A_i, B_i, A'_i, B'_i, Z, and Z' are given by (5.1.6), i.e., they do not depend on the c_i and d_i. Again we have that for all i, $A_i+A'_i=2a_i$ and $B_i+B'_i=2b_i$. Consequently, we have indeed that

$$\sigma|_{\mathfrak{u}(1)^t}=\zeta. \tag{5.3.9}$$

If $\Lambda^*=\sigma|_{(\mathfrak{u}(1)^{p-t}\oplus\mathfrak{u}(1)^{q-t})}$, then

$$\begin{aligned}\Lambda^* &= \left(\frac{r-s}{2}, ..., \frac{r-s}{2}; \frac{s-r}{2}, ..., \frac{s-r}{2}\right)\\ &\quad +(\underbrace{A_1, ..., A_1}_{k_1}, ..., \underbrace{A_x, ..., A_x}_{k_x}, \underbrace{B_1, ..., B_1}_{l_1}, ..., \underbrace{B_y, ..., B_y}_{l_y};\\ &\quad \underbrace{A'_1, ..., A'_1}_{m_1}, ..., \underbrace{A'_x, ..., A'_x}_{m_x}, \underbrace{B'_1, ..., B'_1}_{n_1}, ..., \underbrace{B'_y, ..., B'_y}_{n_y}),\end{aligned}$$

By Frobenius Reciprocity, π_t contains a K-type $\bar{\Lambda}^*$ with highest weight Λ^*. By Proposition 3.2.7, the calculation in the proof of Proposition 5.1.4, and Lemma 3.2.10 we know that $\bar{\Lambda}^*$ and $\bar{\Lambda}$ have the same Vogan norm. But π_t is a limit of discrete series representation, hence has a unique lowest K-type, so we must have $\bar{\Lambda}=\bar{\Lambda}^*$ and $\Lambda=\Lambda^*$, and (5.3.4) is proved.

To check that (5.3.5) holds, notice that

$$\sum_{i=1}^{t}|\zeta_i|=\sum_{i=1}^{x}2c_ia_i-\sum_{i=1}^{y}2d_ib_i. \tag{5.3.10}$$

An easy calculation similar to the one in the proof of (5.2.10) now yields the result. ▮

5.4. *Some More Useful Lemmas*

Using the induction principle, we can use knowledge about the correspondence for representations of $\tilde{U}(p, q)$ to obtain information about the correspondence for representations induced from parabolic subgroups of larger groups with Levi factors of the form $\widetilde{U}(p, q) \times GL(n, \mathbb{C})$. We want to realize an arbitrary representation this way, with the representation on the $\widetilde{U}(p, q)$ factor a discrete series or a limit of discrete series. Recall that for each character $\chi(\zeta, \xi)$ of $(\mathbb{C}^\times)^t$, we defined an irreducible representation $\rho_{\zeta, \xi}$ of $GL(t, \mathbb{C})$ (see Lemma 5.2.6).

LEMMA 5.4.1. *Let $P = MN$ and $P' = M'N'$ be parabolic subgroups of $\widetilde{U}(p, q)$ with Levi factors $M = M_u \times M_{\mathbb{C}^\times}$ and $M' = M_u \times M_{GL}$, where $M_u \cong \widetilde{U}(p-t, q-t)$, $M_{\mathbb{C}^\times} \cong (\mathbb{C}^\times)^t$, and $M_{GL} \cong GL(t, \mathbb{C})$, with $M_{\mathbb{C}^\times} \subset M_{GL}$ and $N' \subset N$. Let σ be a limit of discrete series representation of $\widetilde{U}(p-t, q-t)$, and $\chi = \chi(\zeta, \xi)$ a character of $(\mathbb{C}^\times)^t$. Let π be a lowest K-type constituent of $I = Ind_P^{\widetilde{U}(p, q)}(\sigma \otimes \chi \otimes \mathbb{1})$. Then π is a constituent of $I' = Ind_{P'}^{\widetilde{U}(p, q)}(\sigma \otimes \rho_{\zeta, \xi} \otimes \mathbb{1})$. Moreover, if χ is such that $Re\{\langle \xi, \alpha \rangle\} \geqslant 0$ for all $\alpha \in \Delta(\mathfrak{n})$, then π is a quotient of I'.*

Proof. Since the composition series of I' does not depend on the choice of N', it is sufficient to prove the last assertion. So assume $Re\{\langle \xi, \alpha \rangle\} \geqslant 0$ $\forall \alpha \in \Delta(\mathfrak{n})$, so that π is a quotient of I. Let $P_t = M_{\mathbb{C}^\times} N_t$ be the parabolic subgroup of M_{GL} with $N_t \subset N$. Let $\rho = Ind_{M_{\mathbb{C}^\times} N_t}^{M_{GL}}(\chi \otimes \mathbb{1})$. Since $\mathfrak{n}_t \subset \mathfrak{n}$, $\rho_{\zeta, \xi}$ is the unique irreducible quotient of ρ.

Case 1. $\xi = 0$ and $\zeta_i \in 1 + 2\mathbb{Z}$ for $1 \leqslant i \leqslant t$.
In this case, ρ is irreducible ([D]), hence isomorphic to $\rho_{\zeta, \xi}$, and $I \cong I'$ by Induction by Stages.

Case 2. For every $i \in \{1, 2, ..., t\}$, $\xi_i \neq 0$ or $\zeta_i \in 2\mathbb{Z}$.
Then χ satisfies condition F-2 in [V3] (cf. Section 3), so π is the unique irreducible quotient of I. Therefore, we have a surjective $\widetilde{U}(p, q)$ map $\alpha: I \twoheadrightarrow \pi$, whose kernel is the unique maximal proper $\widetilde{U}(p, q)$ invariant subspace of I. By Induction by Stages, we have that $I \cong Ind_{P'}^{\widetilde{U}(p, q)}(\sigma \otimes \rho \otimes \mathbb{1})$. The surjective M_{GL} map $\rho \twoheadrightarrow \rho_{\zeta, \xi}$ induces a surjective $\widetilde{U}(p, q)$ map $\beta: I \twoheadrightarrow I'$. The kernel of β is either trivial or a proper $\widetilde{U}(p, q)$ invariant subspace of I, hence must be contained in the kernel of α. Consequently, the map α factors through I', and we have a $\widetilde{U}(p, q)$ map $I' \twoheadrightarrow \pi$, i.e., π is a quotient of I'.

Case 3. General χ.
Let m be the number of $i \in \{1, 2, ..., t\}$ such that $\xi_i = 0$ and ζ_i is odd. Let $\chi_1 = \chi((\zeta_{i_1}, ..., \zeta_{i_m}), (\xi_{i_1}, ..., \xi_{i_m}))$ be the corresponding character of $(\mathbb{C}^\times)^m$, which then satisfies the condition of Case 1. Then we may write $\chi = \chi_1 \otimes \chi_2$, where χ_2 is a character of $(\mathbb{C}^\times)^{t-m}$ satisfying the condition of Case 2. This determines $M_1 \times M_2 \subset M_{\mathbb{C}^\times}$ with $M_1 \cong (\mathbb{C}^\times)^m$ and $M_2 \cong (\mathbb{C}^\times)^{t-m}$. The result follows now by first applying Case 1 to χ_1, then

applying Case 2 to χ_2, and using Induction by Stages and the fact that the induction functor preserves direct sums. ∎

The dual pair correspondence for $(GL(t, \mathbb{C}), GL(t, \mathbb{C}))$ is given in Proposition 4.6.1. We need to know how this relates to the results of Lemma 5.2.6.

LEMMA 5.4.2. *For $\sigma \in GL(t, \mathbb{C})^{\frown}$ let $\check{\sigma}$ be as in Proposition* 4.6.1. *If $\sigma \cong \rho_{\zeta, \xi}$, then $\check{\sigma} \cong \rho_{\zeta, -\xi}$.*

Proof. Let $\chi = \chi(\zeta, \xi)$, and $\check{\chi} = \chi(\zeta, -\xi)$. Let $I = Ind_{P_t}^{GL(t, \mathbb{C})}(\chi \otimes \mathbb{1})$, and $I' = Ind_{P_t}^{GL(t, \mathbb{C})}(\check{\chi} \otimes \mathbb{1})$, where P_t is as in the proof of Lemma 5.4.1. Let Θ_I and $\Theta_{I'}$ be the global characters of I and I' respectively, given as functions on the regular semisimple elements of $G = GL(t, \mathbb{C})$. A direct calculation using the induced character formula (Theorem 5.7 of [HS]) yields that $\Theta_{I'}(g) = \Theta_I(\bar{g}^{-1})$ for all regular semisimple elements g of G. Also, $\Theta_{\check{\sigma}}(g) = \Theta_\sigma(\bar{g}^{-1})$. It follows that $\check{\rho}_{\zeta, \xi}$ is a constituent of I'. Since $\rho_{\zeta, \xi}$ contains the K-type $\bar{\zeta}$ with highest weight Weyl group conjugate to ζ, so does $\check{\rho}_{\zeta, \xi}$. But $\bar{\zeta}$ is of multiplicity one in I', and is contained in $\rho_{\zeta, -\xi}$, so $\check{\rho}_{\xi, \zeta} \cong \rho_{\zeta, -\xi}$. ∎

LEMMA 5.4.3. *The lowest K-type of $\rho_{\zeta, \xi}$ is of minimal degree and corresponds to the lowest K-type of $\rho_{\zeta, -\xi}$ in the space of joint harmonics for the dual pair $(GL(t, \mathbb{C}), GL(t, \mathbb{C}))$.*

Proof. This follows from Lemma 4.1 of [AB1] and Proposition 4.6.3. ∎

LEMMA 5.4.4. *In the setting of Theorem* 4.6.4, *there exist characters α_1 and α_2 of M_1 and M_2, trivial on $K_1 \cap M_1$ and $K_2 \cap M_2$, such that $Ind_{P_1}^{G_1}(\pi \otimes \sigma \otimes \alpha_1 \otimes \mathbb{1})$ is irreducible, and $(\pi \otimes \sigma \otimes \alpha_1) \otimes (\pi' \otimes \check{\sigma} \otimes \alpha_2)$ is a quotient of ω_M.*

Proof. Let $z \in \mathbb{C}$, and let α_1 and α_2 be the characters of M_1 and M_2 given by

$$\begin{aligned} \alpha_1(u, g) &= |det(g)|^z && \text{for} \quad u \in \widetilde{U}(p, q), \quad g \in GL(k, \mathbb{C}), \\ \alpha_2(v, h) &= |det(h)|^{-z} && \text{for} \quad v \in \widetilde{U}(r, s), \quad h \in GL(k, \mathbb{C}). \end{aligned} \tag{5.4.5}$$

Then α_i is trivial on $K_i \cap M_i$ for $i = 1, 2$, and since

$$\check{\sigma}\, |det|^{-z} = (\sigma\, |det|^z)^\vee,$$

we have that $(\pi \otimes \sigma \otimes \alpha_1) \otimes (\pi' \otimes \check{\sigma} \otimes \alpha_2)$ is a quotient of ω_M. So the lemma follows if we can find $z \in \mathbb{C}$ such that $Ind_{P_1}^{G_1}(\pi \otimes \sigma \otimes |det|^z \otimes \mathbb{1})$ is irreducible. This is well known and follows from an argument similar to one used in [SV]. ∎

6. THE CORRESPONDENCE

THEOREM 6.1. *Let $\widetilde{U}(p, q)$ be as in Section 3, and let $\pi \in \widetilde{U}(p, q)\widehat{\ }_{genuine}$. Then $\theta_{r,s}(\pi) = \pi'$, where r, s, and π' are given as follows*:

(a) *If π is the discrete series representation with Harish–Chandra parameter λ, then $r = r(\lambda)$, $s = s(\lambda)$, and π' is the discrete series representation with HC parameter $\lambda' = \Gamma(\lambda)$ (see Definition 5.2.2).*

(b) *If $\pi = \pi(\lambda, \Psi)$ is a genuine limit of discrete series representation of $\widetilde{U}(p, q)$, then $r = r(\lambda)$, $s = s(\lambda)$, and π' is the limit of discrete series representation of $\widetilde{U}(r, s)$ given by the parameters $(\Gamma\lambda, \Gamma\Psi)$.*

(c) *If $\pi = \bar{I}(\gamma)$ with $M \cong \widetilde{U}(p-k, q-k) \times (\mathbb{C}^{\times})^k$ and $I(\gamma) = Ind_{MN}^{\widetilde{U}(p,q)}(\sigma \otimes \chi \otimes \mathbb{1})$ as in Section 3, let r', s' and the limit of discrete series representation σ' of $\widetilde{U}(r', s')$ be obtained from the limit of discrete series representation σ of $\widetilde{U}(p-k, q-k)$ as in (b). Then $r = r' + k$, $s = s' + k$, and $\pi' = \bar{I}(\gamma')$ with $M' \cong \widetilde{U}(r-k, s-k) \times (\mathbb{C}^{\times})^k$ and $I(\gamma') = Ind_{M'N'}^{\widetilde{U}(r,s)}(\sigma' \otimes \chi \otimes \mathbb{1})$.*

Moreover, if μ is a lowest K-type of π, then μ is of minimal r, s-degree in π, and μ corresponds to a lowest K-type of π' in the space of joint harmonics.

Proof. (a) This follows from Theorem 6.2 of [L].

(b) Let $\{a_{i_1}, ..., a_{i_{x_0}}\}$ list the a_i with the property that $k_i > m_i$, $\{b_{i_1}, ..., b_{i_{y_0}}\}$ the b_i with the property that $l_i > n_i$, $\{a_{i_{x_0+1}}, ..., a_{i_{x_0+z_0}}\}$ the a_i with the property that $m_i > k_i$, and $\{b_{i_{y_0+1}} ..., b_{i_{y_0+w_0}}\}$ the b_i with the property that $n_i > l_i$. Let $p_0 = x_0 + y_0$, $q_0 = z_0 + w_0$, and $t = p - p_0 = q - q_0$.

Let $\pi_0 = \pi_0(\lambda_0)$ be the discrete series representation of $\widetilde{U}(p_0, q_0)$ with Harish–Chandra parameter

$$\lambda_0 = (a_{i_1}, a_{i_2}, ..., a_{i_{x_0}}, b_{i_1}, ..., b_{i_{y_0}}; a_{i_{x_0+1}}, ..., a_{i_{x_0+z_0}}, b_{i_{y_0+1}} ..., b_{i_{y_0+w_0}}). \tag{6.2}$$

By part (a), $\theta_{r_0, s_0}(\pi_0) = \pi'_0$, where $r_0 = x_0 + w_0$, $s_0 = y_0 + z_0$, and π'_0 is the discrete series representation of $\widetilde{U}(r_0, s_0)$ with Harish–Chandra parameter

$$\lambda'_0 = (a_{i_1}, a_{i_2}, ..., a_{i_{x_0}}, b_{i_{y_0+1}}, ..., b_{i_{y_0+\omega_0}}; a_{i_{x_0+1}}, ..., a_{i_{x_0+z_0}}, b_{i_1} ..., b_{i_{y_0}}). \tag{6.3}$$

Notice that $r = r_0 + t$ and $s = s_0 + t$. Let $\chi = \chi(\zeta, \xi)$ be the character of $(\mathbb{C}^{\times})^t$ given by

$$\zeta = (\underbrace{2a_1, ..., 2a_1}_{\min\{k_1, m_1\} \text{ terms}}, ..., \underbrace{2a_x, ..., 2a_x}_{\min\{k_x, m_x\} \text{ terms}}, \underbrace{2b_1, ..., 2b_1}_{\min\{l_1, n_1\} \text{ terms}}, ..., \underbrace{2b_y, ..., 2b_y}_{\min\{l_y, n_y\} \text{ terms}}), \tag{6.4}$$

and $\xi = (0, ..., 0)$.

By Lemma 3.2.6 and Lemma 5.4.1, π is a direct summand of

$$I = Ind_P^{\widetilde{U}(p,q)}(\pi_0 \otimes \rho_{\zeta,\xi} \otimes \mathbb{1}), \tag{6.5}$$

where $P=MN$ is any parabolic subgroup of $\tilde{U}(p,q)$ with Levi factor $M\cong\tilde{U}(p_0,q_0)\times GL(t,\mathbb{C})$, and $\rho_{\zeta,\xi}$ as defined in Lemma 5.2.6. Similarly, π' is a direct summand of

$$I'=Ind_{P'}^{\tilde{U}(r,s)}(\pi_0'\otimes\rho_{\zeta,\xi}\otimes\mathbb{1}),\tag{6.6}$$

for $P'=M'N'$ with $M'\cong\tilde{U}(r_0,s_0)\times GL(t,\mathbb{C})$.

Let ω be the oscillator representation for the dual pair $(U(p,q),U(r,s))$. Since $\xi=0$, we have by Lemma 5.4.2 that $\check{\rho}_{\zeta,\xi}=\rho_{\zeta,\xi}$, thus by Theorem 4.6.4, there is a nonzero $\tilde{U}(p,q)\times\tilde{U}(r,s)$ map

$$\phi\colon\omega\to Ind_P^{\tilde{U}(p,q)}(\pi_0\otimes\rho_{\zeta,\xi}\otimes\mathbb{1})\otimes Ind_{P'}^{\tilde{U}(r,s)}(\pi_0'\otimes\rho_{\zeta,\xi}\otimes\mathbb{1})\tag{6.7}$$

for some choice of parabolic subgroups $P=MN$ and $P'=M'N'$ wit M and M' as in (6.5) and (6.6). Let μ and μ' be the unique lowest K-types of π and π' respectively. Then μ and μ' have multiplicity one in I and I'. By Proposition 5.1.4 and Lemma 5.2.5, μ and μ' are of minimal degree in I and I' and correspond in the space of joint harmonics for the dual pair $(U(p,q),U(r,s))$. Consequently, if we show that $\mu\otimes\mu'$ is in the image of ϕ, then by irreducibility we must have that $\pi\otimes\pi'$ is in the image of ϕ. Since $\pi\otimes\pi'$ is a direct summand of $I\otimes I'$, this implies $\theta_{r,s}(\pi)=\pi'$.

We use Theorem 4.6.6, with $\mu_1=\mu$ and $\mu_2=\mu'$. Let μ_0 be the lowest K-type of π_0, $\bar{\zeta}$ the lowest K-type of $\rho_{\zeta,\xi}$, and let $\eta=\mu_0\otimes\bar{\zeta}$, a K-type for $M=\tilde{U}(p_0,q_0)\times GL(t,\mathbb{C})$. By part (a) and Lemma 5.4.3, η is of minimal degree in $\pi_0\otimes\rho_{\zeta,\xi}$ (for the dual pair $(U(p_0,q_0)\times GL(t,\mathbb{C}),\ U(r_0,s_0)\times GL(t,\mathbb{C}))$). By Lemma 5.2.8, μ and η have the same degree (each degree w.r.t. the appropriate setting), and the restriction of μ to $M\cap K$ contains η. Consequently, the assumptions (1), (2), and (3) of Theorem 4.6.6 are satisfied. Condition (4) follows from Lemma 5.4.4., so that $\mu\otimes\mu'$ is in the image of ϕ, and assertion (b) is proved.

(c) Let ω be the oscillator representation for the dual pair $(U(p,q),U(r,s))$. By Theorem 4.6.4, there exist parabolic subgroups $P=MN$ and $P'=M'N'$ of $\tilde{U}(p,q)$ and $\tilde{U}(r,s)$ with Levi factors $M\cong\tilde{U}(p-k,q-k)\times GL(k,\mathbb{C})$ and $M'\cong\tilde{U}(r-k,s-k)\times GL(k,\mathbb{C})$ such that for any $\rho\in GL(k,\mathbb{C})^\frown$, there is a nonzero $\tilde{U}(p,q)\times\tilde{U}(r,s)$ map

$$\phi\colon\omega\to Ind_P^{\tilde{U}(p,q)}(\sigma\otimes\rho\otimes\mathbb{1})\otimes Ind_{P'}^{\tilde{U}(r,s)}(\sigma'\otimes\check{\rho}\otimes\mathbb{1}).\tag{6.8}$$

If $\chi=\chi(\zeta,\xi)$ with $\xi=(\xi_1,...,\xi_k)$, let $\rho=\rho_{\zeta,\xi}$. By Lemma 5.4.1, π is the unique lowest K-type constituent of $I=Ind_P^{\tilde{U}(p,q)}(\sigma\otimes\rho\otimes\mathbb{1})$. We may assume that π is a quotient of I: if necessary, change the order of the (ζ_i,ξ_i) and replace some of the ξ_i by $-\xi_i$ to guarantee that condition (3.1.6) is satisfied (see Section 3.1).

Let μ be a lowest K-type of π. Then μ is a lowest K-type of I of multiplicity one. By Lemma 5.4.1 and Proposition 5.1.4, μ is r, s-harmonic and of minimal r, s-degree in I. By Lemmas 5.4.1 and 3.2.6, there exists a standard representation I'_d induced from discrete series which contains $I' = Ind_{P'}^{\tilde{U}(r,s)}(\sigma' \otimes \check{\rho} \otimes \mathbb{1})$ as a summand, and we have that every lowest K-type of I' is a lowest K-type of I'_d, and every lowest K-type of I'_d which occurs in I' is a lowest K-type of I'. By Lemma 5.3.1, μ corresponds to a minimal K-type μ' of I'_d in the space of joint harmonics for the dual pair $(U(p, q), U(r, s))$.

Claim 6.9. *The image of ϕ contains $\mu \otimes \mu'$.*

Assuming the claim, we have that μ' is a (lowest) K-type of I'. By Lemma 5.4.1, I' has a unique constituent containing μ'. By Lemma 5.4.2, this constituent is isomorphic to $\bar{I}(\gamma') = \pi'$. Since π is an irreducible quotient of I, we have a nonzero $\tilde{U}(p, q) \times \tilde{U}(r, s)$ map $\omega \to \pi \otimes I'$, and it follows that π corresponds to the unique constituent π' of I' containing μ', and (c) is proved. The proof of the claim is similar to (b); we use Theorem 4.6.6, Proposition 5.1.4, and Lemmas 5.4.3, 5.3.3, and 5.4.4. ∎

Remark 6.10. For the special case $p = q = k$ in part (c) of Theorem 6.1, we formally define the oscillator representation of $\widetilde{Sp}(0, \mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ to be the nontrivial character *sgn*, so that the correspondence for the dual pair $(U(0, 0), U(0, 0))$ is given by $\mathbb{1} \leftrightarrow \mathbb{1}$.

7. REPRESENTATIONS OF THE FORM $A_{\mathfrak{q}}(\lambda)$

Let $G = \tilde{U}(p, q)$ with complexified Lie algebra $\mathfrak{g}$ as in Section 3. According to Theorem 6.1, a discrete series representation of G corresponds to a discrete series representation of some $\tilde{U}(r, s)$ of the same rank, and limits of discrete series correspond to limits of discrete series. We now consider representations of G of the form $A_{\mathfrak{q}}(\lambda)$ which are "good" in the sense of Definition 0.49 of [KV]. Once we know the Langlands parameters of such representations, Theorem 6.1 will enable us to determine their image under the correspondence in the equal rank case.

For $\lambda \in i(\mathfrak{t}_0)^*$ with $\langle \lambda, \alpha \rangle \in \frac{1}{2}\mathbb{Z}$ for all roots α of $\mathfrak{g}$, let $\mathfrak{q} = \mathfrak{q}(\lambda) = \mathfrak{l} + \mathfrak{u}$ be the theta stable parabolic subalgebra of $\mathfrak{g}$ defined by λ as in Proposition 3.2.7. Let $A^*(\lambda) = A_{\mathfrak{q}}(\lambda - \rho(\mathfrak{u}))$, where $A_{\mathfrak{q}}(\cdot)$ is as in Chapter V of [KV]. Recall that $A^*(\lambda)$ has infinitesimal character $\lambda + \rho(\mathfrak{l})$. Write

$$\lambda = (\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{a_1, ..., a_1}_{l_1}, ..., \underbrace{a_x, ..., a_x}_{l_x}) \tag{7.1}$$

with $a_1 > a_2 > \cdots > a_x$.

LEMMA 7.2. *If λ satisfies*

$$a_i \in \mathbb{Z} + \frac{k_i + m_i}{2} \qquad \text{for} \quad 1 \leqslant i \leqslant x, \tag{7.3a}$$

and

$$a_i - a_{i+1} \geqslant \frac{k_i + l_i + k_{i+1} + l_{i+1}}{2} \qquad \text{for} \quad 1 \leqslant i \leqslant x-1, \tag{7.3b}$$

then $A^*(\lambda)$ is a non-zero genuine irreducible unitary representation of G, with regular infinitesimal character and unique lowest K-type with highest weight $\lambda + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$.

Proof. The lemma follows from [VZ]. ∎

PROPOSITION 7.4 (Langlands parameters of $A_{\mathfrak{q}}(\lambda)$'s). *Let λ and $\pi = A^*(\lambda)$ be as in Lemma 7.2. Then π is the unique lowest K-type constituent of the standard representation*

$$I = Ind_{MN}^{\tilde{U}(p,\,q)}(\sigma \otimes \chi \otimes \mathbb{1}) \tag{7.5}$$

given by the following data.

For $i = 1, 2, ..., x$, let $m_i = \min\{k_i, l_i\}$, $k'_i = k_i - m_i$, and $l'_i = l_i - m_i$. (Notice that for all i, either k'_i or l'_i is zero.) Let $m = \sum_{i=1}^{x} m_i$. Then $M \cong \tilde{U}(p-m, q-m) \times (\mathbb{C}^\times)^m$ and $\sigma = \sigma(\lambda_M, \Psi_M)$ is the discrete series representation of $\tilde{U}(p-m, q-m)$ with HC parameter.

$$\begin{aligned}\lambda_M = (&\alpha_{11}, \alpha_{12}, ..., \alpha_{1k'_1}, \alpha_{21}, ..., \alpha_{2k'_2}, ..., \alpha_{x1}, ..., \alpha_{xk'_x};\\ &\alpha'_{11}, \alpha'_{12}, ..., \alpha'_{1l'_1}, \alpha'_{21}, ..., \alpha'_{2l'_2}, ..., \alpha'_{x1}, ..., \alpha'_{xl'_x}).\end{aligned} \tag{7.6a}$$

Here

$$\begin{aligned}(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{ik'_i}) &= (a_i, a_i, ..., a_i) + \rho(k'_i)\\ (\alpha'_{i1}, \alpha'_{i2}, ..., \alpha'_{il'_i}) &= (a_i, a_i, ..., a_i) + \rho(l'_i)\end{aligned} \tag{7.6b}$$

with $\rho(n)$ for a non-negative integer n defined by $\rho(n) = ((n-1)/2, (n-3)/2, ..., (-n+1)/2)$. The character χ of $(\mathbb{C}^\times)^m$ is given by

$$\chi = \bigotimes_{i=1}^{x} \chi_i, \tag{7.7a}$$

where $\chi_i = \chi(\mu_i, \nu_i)$ *is the character of* $(\mathbb{C}^\times)^{m_i}$ *with*

$$\mu_i = (2a_i, ..., 2a_i) \quad \textit{and} \quad \nu_i = (k_i + l_i - 1, k_i + l_i - 3, ..., k_i + l_i - 2m_i + 1). \tag{7.7b}$$

Proof. This follows from Corollary 11.219 of [KV]. ∎

THEOREM 7.8. *Let* $\pi \cong A^*(\lambda) \in \widetilde{U}(p, q)\widehat{}_{genuine}$ *with*

$$\lambda = (\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{l_1}, ..., \underbrace{b_y, ..., b_y}_{l_y};$$
$$\underbrace{a_1, ..., a_1}_{m_1}, ..., \underbrace{a_x, ..., a_x}_{m_x}, \underbrace{0, ..., 0}_{z}, \underbrace{b_1, ..., b_1}_{n_1}, ..., \underbrace{b_y, ..., b_y}_{n_y}), \tag{7.9}$$

$a_1 > \cdots > a_x > 0 > b_1 > \cdots > b_y$, *and* $k_i + m_i \neq 0 \neq l_j + n_j$ *for all* i, j. *Assume that* λ *satisfies the conditions of Lemma 7.2 so that* π *has a regular infinitesimal character. Assume in addition that*

$$\textit{either} \qquad w = z \neq 0,$$
$$\textit{or} \qquad \begin{cases} w = z = 0 \quad \textit{and} \\ a_x \geqslant \left|\dfrac{k_x - m_x}{2}\right| \quad (\textit{if } x \neq 0) \quad \textit{and} \\ -b_1 \geqslant \left|\dfrac{l_1 - n_1}{2}\right| \quad (\textit{if } y \neq 0). \end{cases} \tag{7.10}$$

Let $r = \sum_{i=1}^{x} k_i + \sum_{i=1}^{y} n_i + w$ *and* $s = \sum_{i=1}^{x} m_i + \sum_{i=1}^{y} l_i + w$.

Then $\theta_{r,s}(\pi) = A^*(\lambda')$, *where*

$$\lambda' = (\underbrace{a_1, ..., a_1}_{k_1}, \underbrace{a_2, ..., a_2}_{k_2}, ..., \underbrace{a_x, ..., a_x}_{k_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{n_1}, ..., \underbrace{b_y, ..., b_y}_{n_y};$$
$$\underbrace{a_1, ..., a_1}_{m_1}, ..., \underbrace{a_x, ..., a_x}_{m_x}, \underbrace{0, ..., 0}_{w}, \underbrace{b_1, ..., b_1}_{l_1}, ..., \underbrace{b_y, ..., b_y}_{l_y}). \tag{7.11}$$

Proof. This follows from Theorem 6.1 after computing the Langlands parameters of π using Proposition 7.4. Let λ_M be the Harish–Chandra parameter obtained from λ as in Proposition 7.4. Since λ satisfies condition (7.3b), (7.10) implies $a_i \geqslant |(k_i - m_i)/2|$ and $-b_j \geqslant |(l_j - n_j)/2|$ for all i, j. This means that all coefficients of

$$(\underbrace{a_i, ..., a_i}_{|k_i - m_i|}) + \rho(|k_i - m_i|)$$

are strictly positive, and all coefficients of

$$\underbrace{(b_j, ..., b_j)}_{|l_j - n_j|} + \rho(|l_j - n_j|)$$

are strictly negative (see (7.6)). Also, $w = z$ implies that the 0's do not appear in λ_M. Using Theorem 6.1, we have that $\theta_{r,s}(\pi) \neq 0$ for r, s given above, and the Langlands parameters of $\theta_{r,s}(\pi)$ are again of the form given in Proposition 7.4., for λ' given by (7.11). ∎

In general, a representation π may be realized as an $A^*(\lambda)$ in more than one way. Clearly, π will correspond to an $A^*(\lambda')$ if for *one* of the possible realizations, λ satisfies the conditions of Theorem 7.8. However, we should have the following

CONJECTURE 7.12. *Let π be of the form $A^*(\lambda)$ and satisfy the conditions of Lemma 7.2. Assume that for* every *realization $A^*(\lambda_i)$ of π, λ_i fails to satisfy condition* (7.10), *and that $\theta_{r,s}(\pi) = \pi'$ for some r, s with $r + s = p + q$. Then π' is not of the form $A^*(\lambda')$.*

Remark 7.13. Since π' has regular integral infinitesimal character, Theorem 1.2 of [SR] implies that under the conditions of Conjecture 7.12, π' is not unitary.

EXAMPLE 7.14. Let $p + q = 2n$, and let $\mathbb{1}_{p,q}$ be the trivial representation of $U(p, q)$, an $A_{\mathfrak{q}}(\lambda)$ with regular infinitesimal character. Let $m = |(p-q)/2|$. We have that $\theta_{n,n}(\mathbb{1}_{p,q}) = \pi'$, where π' is the unique lowest K-type constituent of

$$Ind_{(U(m, m) \times (\mathbb{C}^\times)^{n-m})\,N}^{U(n, n)}(\sigma \otimes \chi(0, v) \otimes \mathbb{1}).$$

Here σ is the discrete series of $U(m, m)$ with HC parameter

$$\begin{cases} (m - \frac{1}{2}, m - \frac{3}{2}, ..., \frac{1}{2}; -\frac{1}{2}, ..., -m + \frac{1}{2}) & \text{if} \quad p > q, \\ (-\frac{1}{2}, -\frac{3}{2}, ..., -m + \frac{1}{2}; m - \frac{1}{2}, ..., \frac{1}{2}) & \text{if} \quad p < q; \end{cases}$$

and $v = (2m+1, 2m+3, ..., 2n-1)$. The lowest K-type of π' has weight $((p-q)/2, ..., (p-q)/2;\ (q-p)/2, ..., (q-p)/2)$, and it is easy to check (using Prop. 8.6 of [SR]) that unless $p = q$ or $pq = 0$, π' is *not* an $A_{\mathfrak{q}}(\lambda)$, and hence not unitary. In the case $p = q$, $\pi' = \mathbb{1}_{p,p}$, and if $pq = 0$, π' is a discrete series representation.

The following proposition gives, in a special case, some more information about the representation an $A_{\mathfrak{q}}(\lambda)$ corresponds to in the case where λ fails to satisfy (7.10).

PROPOSITION 7.15. *Let k be a positive integer. Let*

$$(a_1, a_2, ..., a_x, b_1, ..., b_y; c_1, ..., c_z, d_1, ..., d_w) \tag{7.16}$$

be the Harish–Chandra parameter of a genuine discrete series representation of $\tilde{U}(p, q)$ with a_x, c_z, $-b_1$, $-d_1 \geqslant k+\frac{1}{2}$. For $-k \leqslant l \leqslant k$, consider the representations $\pi_l = A^*(\lambda_l)$ of $G_l = \tilde{U}(p+k+l, q+k-l)$ given by

$$\lambda_l = (a_1, ..., a_x, \underbrace{0, ..., 0}_{k+l}, b_1, ..., b_y; c_1, ..., c_z, \underbrace{0, ..., 0}_{k-l}, d_1, ..., d_w). \tag{7.17}$$

Let $r = x+w$ and $s = y+z$. Then $\theta_{r+k,s+k}(\pi_l) \neq 0$ for all l. Moreover, if we let $\pi'_l = \theta_{r+k,s+k}(\pi_l)$, then π'_l is a constituent of the standard representation I of $G' = \tilde{U}(r+k, s+k)$ whose unique Langlands quotient is isomorphic to $\pi'_0 = A^*(\lambda')$ given by

$$\lambda' = (a_1, ..., a_x, \underbrace{0, ..., 0}_{k}, d_1, ..., d_w; c_1, ..., c_z, \underbrace{0, ..., 0}_{k}, b_1, ..., b_y). \tag{7.18}$$

Proof. Let $\rho' \in \tilde{U}(r, s)\widehat{\ }_{genuine}$ be the discrete series representation with HC parameter

$$(a_1, a_2, ..., a_x, d_1, ..., d_w; c_1, ..., c_z, b_1, ..., b_y). \tag{7.19}$$

By Theorem 6.2 of [L], $\theta_{r,s}(\pi_l) = \rho'$. Let ω and ω_M be the oscillator representations for the dual pairs (G_l, G') and $(G_l, \tilde{U}(r, s))$ respectively. We have a non-zero map $\omega_M \twoheadrightarrow \pi_l \otimes \rho'$. By Theorem 4.5.5, there is a non-zero $G_l \times G'$ map $\omega \to \pi_l \otimes I'$, where $I' = Ind_P^{G'}(\rho' \otimes |det|^{-k} \otimes \mathbb{1})$ for $P = MN$ a parabolic subgroup of G' with Levi factor $M \cong \tilde{U}(r, s) \times GL(k, \mathbb{C})$. Consequently, $\theta_{r+k,s+k}(\pi_l)$ is non-zero and a constituent of I'. The character $|det|^{-k}$ of $GL(k, \mathbb{C})$ is isomorphic to $\rho_{0,\xi}$ with

$$\xi = (k-1, k-3, ..., -k+1) + (-k, -k, ..., -k) = (-1, -3, ..., -2k+1).$$

By Proposition 7.4, $I \cong Ind_{P'}^{G'}(\rho' \otimes \chi_{0,-\xi} \otimes \mathbb{1})$ with $P' = M'N'$ and $M' \cong \tilde{U}(r, s) \times (\mathbb{C}^\times)^k$, and by the discussion in Chapter 3, I has the same constituents as $Ind_{P'}^{G'}(\rho' \otimes \chi_{0,\xi} \otimes \mathbb{1})$. Using Induction by Stages, we see that every irreducible constituent of I' is an irreducible constituent of I, and the proposition is proved. ∎

EXAMPLE 7.14 (continued; see also [Z]). For every p and q with $p+q = 2n$, $\theta_{n,n}(\mathbb{1}_{p,q})$ is a constituent of

$$Ind_{(\mathbb{C}^\times)^n N}^{U(n,n)}(\chi(0, v) \otimes \mathbb{1}),$$

where $v = (1, 3, ..., 2n-1)$.

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